Potential Theory and Nonlinear Elliptic Equations
Lecture 6

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Publications

1. I. Verbitsky, *Bilateral estimates of solutions to quasilinear elliptic equations with sub-natural growth terms*, ***Adv. Calc. Var.*** (published online, April 2021), DOI: 10.1515/acv-2021-0004

2. Nguyen Cong Phuc and I. Verbitsky, *Singular quasilinear and Hessian equations and inequalities*, ***J. Funct. Analysis***, **256** (2009) 1875–1906.

3. Dat Tien Cao and I. Verbitsky, *Nonlinear elliptic equations and intrinsic potentials of Wolff type*, ***J. Funct. Analysis***, **272** (2017) 112–165.

4. Dat Tien Cao and I. Verbitsky, *Pointwise estimates of Brezis–Kamin type for solutions of sublinear elliptic equations*, ***Nonlin. Analysis***, **146** (2016) 1–19.

Additional literature

1. D. R. Adams and L. I. Hedberg, *Function Spaces and Potential Theory*, Grundlehren der math. Wissenschaften, **314**, Springer, Berlin, 1996.
2. T. Kilpeläinen and J. Malý, *The Wiener test and potential estimates for quasilinear elliptic equations*, Acta Math., **172** (1994), 137–161.
3. J. Malý and W. Ziemer, *Fine Regularity of Solutions of Elliptic Partial Differential Equations*, Math. Surveys Monogr., **51**, Amer. Math. Soc., Providence, RI, 1997.
4. V. G. Maz'ya, *Sobolev Spaces, with Applications to Elliptic Partial Differential Equations*, 2nd revised augm. ed., Grundlehren der math. Wissenschaften, **342**, Springer, Berlin, 2011.

Homogeneous integral equations with Wolff potentials

Let $1 < p < \infty$, $0 < \alpha < \frac{n}{p}$, and $0 < q < p-1$. Fix $\sigma \in \mathcal{M}^+(\mathbb{R}^n)$. In Lecture 5, we studied the integral equation with $\mathbf{W} = \mathbf{W}_{\alpha,p}$,

$$u(x) = \mathbf{W}(u^q d\sigma)(x), \quad u \ge 0, \quad x \in \mathbb{R}^n. \tag{1}$$

Recall that equation (1) is understood $d\sigma$-a.e., and $u < \infty$ $d\sigma$-a.e., or equivalently $u \in L^q_{\rm loc}(\mathbb{R}^n, \sigma)$. We can always choose a representative which coincides with u $d\sigma$-a.e., defined for all $x \in \mathbb{R}^n$, such that (1) is understood everywhere in $\mathbb{R}^n$.

We also considered the corresponding **subsolutions** $u \ge 0$ such that

$$u(x) \le \mathbf{W}(u^q d\sigma)(x) < \infty, \quad x \in \mathbb{R}^n, \tag{2}$$

and **supersolutions** $u \ge 0$ such that

$$\mathbf{W}(u^q d\sigma)(x) \le u(x) < \infty, \quad x \in \mathbb{R}^n. \tag{3}$$

Integral equations with Wolff potentials

For any $\nu \in \mathcal{M}^+(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$ so that $\mathbf{W}\nu(x) < \infty$, we set

$$\phi_\nu(x) := \mathbf{W}\nu(x) \left(\frac{\mathbf{W}[(\mathbf{W}\nu)^q d\sigma](x)}{\mathbf{W}\nu(x)} \right)^{\frac{p-1}{p-1-q}}, \tag{4}$$

$$\phi(x) := \sup\{\phi_\nu(x) : \ \nu \in \mathcal{M}^+(\mathbb{R}^n), \ \mathbf{W}\nu(x) < \infty\}. \tag{5}$$

In Lecture 5, we stated the following theorem (a proof is given below).

Theorem 25 (Verbitsky 2021)

Any nontrivial solution u to (1) satisfies the following estimates,

$$C\,\phi(x) \le u(x) \le \phi(x), \qquad x \in \mathbb{R}^n, \tag{6}$$

with positive constant $C = C(\alpha, p, q, n)$. Moreover, the upper bound holds for any **subsolution** u*, whereas the lower bound in holds for any nontrivial* **supersolution** u.

Integral equations with Wolff potentials

In Lecture 5, we also proved the following three lemmas.

Lemma 1

Let $1 < p < \infty$, $0 < \alpha < \frac{n}{p}$, *and* $0 < q < p-1$. *Let* $\sigma \in \mathcal{M}^+(\mathbb{R}^n)$. *Suppose* u *is a subsolution to* (1). *Then*

$$u(x) \le \phi(x), \qquad x \in \mathbb{R}^n, \tag{7}$$

provided $u(x) \le \mathbf{W}(u^q d\sigma)(x) < \infty$. *In paticular,* (7) *holds* $d\sigma$-*a.e.*

Lemma 2

Let $\nu, \sigma \in \mathcal{M}^+(\mathbb{R}^n)$. *Then for* $C = C(\alpha, p, q, n) > 0$,

$$\begin{aligned} \mathbf{W}[(\mathbf{W}\nu)^q d\sigma](x) &\le C\,(\mathbf{W}\nu(x))^{\frac{q}{p-1}} \\ &\times \left[\mathbf{W}\sigma(x) + (\mathbf{K}\sigma(x))^{\frac{p-1-q}{p-1}}\right], \quad x \in \mathbb{R}^n. \end{aligned} \tag{8}$$

Integral equations with Wolff potentials

Lemma 3

Let $1 < p < \infty$, $0 < \alpha < \frac{n}{p}$, *and* $0 < q < p - 1$. *Let* $\sigma \in \mathcal{M}^+(\mathbb{R}^n)$. *Then there exist constants* $C_i = C_i(\alpha, p, q, n) > 0$ *(*$i = 1, 2$*) so that*

$$C_1\,\phi(x) \le (\mathbf{W}\sigma(x))^{\frac{p-1}{p-1-q}} + \mathbf{K}\sigma(x) \le C_2\,\phi(x), \tag{9}$$

where the lower estimate holds for all $x \in \mathbb{R}^n$, *whereas the upper estimate holds provided* $\mathbf{W}\sigma(x) < \infty$ *and* $\mathbf{K}\sigma(x) < \infty$.
If $\mathbf{W}\sigma \not\equiv \infty$ *and* $\mathbf{K}\sigma \not\equiv \infty$, *then* $\phi < \infty$ $d\sigma$*-a.e., and the upper estimate in* (9) *holds* $d\sigma$*-a.e.*

We are now ready to complete the proof of Theorem 25.

Integral equations with Wolff potentials

Proof of Theorem 25. The upper bound in (6) follows from Lemma 1.

The lower bound in Theorem 25 is a consequence of Lemma 3 and the following lower estimate [Cao-V. 2017], for any nontrivial supersolution u and a positive constant $C = C(\alpha, p, q, n)$,

$$C \left[(\mathbf{W}\sigma(x))^{\frac{p-1}{p-1-q}} + \mathbf{K}\sigma(x) \right] \le u(x), \quad x \in \mathbb{R}^n. \tag{10}$$

The proof of estimate (10) is split into two parts:

$$\textbf{(A)} \qquad C \, (\mathbf{W}\sigma(x))^{\frac{p-1}{p-1-q}} \le u(x), \quad \forall x \in \mathbb{R}^n, \tag{11}$$

$$\textbf{(B)} \qquad C \, \mathbf{K}\sigma(x) \le u(x), \quad \forall x \in \mathbb{R}^n. \tag{12}$$

Proof of the lower bound (A)

We will need the following lemma (an analogue of the integration by parts lemma in Lecture 3).

Lemma 4 (iterated Wolff potentials)

Let $1 < p < \infty$, $0 < \alpha < \frac{n}{p}$, $\sigma \in \mathcal{M}^+(\mathbb{R}^n)$. Then, for all $r > 0$,

$$\mathfrak{c}^{\frac{r}{p-1}} \left(\mathbf{W}\sigma(x)\right)^{\frac{r}{p-1}+1} \le \mathbf{W}\left[(\mathbf{W}\sigma)^r d\sigma\right](x), \quad x \in \mathbb{R}^n, \qquad (13)$$

where $\mathfrak{c} = \mathfrak{c}(\alpha, p, n)$ is a positive constant (which does not depend on r).

Proof of Lemma 4. For $t > 0$, obviously,

$$\mathbf{W}\sigma(y) = \int_0^t \left(\frac{\sigma(B(y,s))}{s^{n-\alpha p}}\right)^{\frac{1}{p-1}} \frac{ds}{s} + \int_t^\infty \left(\frac{\sigma(B(y,s))}{s^{n-\alpha p}}\right)^{\frac{1}{p-1}} \frac{ds}{s}.$$

Proof of Lemma 4

By the lemma on Wolff potentials in Lecture 5, there exists a positive constant $C = C(p, \alpha, n)$ so that, for any ball $B = B(x, t)$,

$$\inf_{B(x,t)} \mathbf{W}\sigma \geq C \int_t^\infty \left(\frac{\sigma(B(x,s))}{s^{n-\alpha p}} \right)^{\frac{1}{p-1}} \frac{ds}{s}. \tag{14}$$

Notice that, for the iterated Wolff potential we have

$$\mathbf{W}[(\mathbf{W}\sigma)^r d\sigma](x) = \int_0^\infty \left(\frac{\int_{B(x,t)} [\mathbf{W}\sigma(y)]^r d\sigma(y)}{t^{n-\alpha p}} \right)^{\frac{1}{p-1}} \frac{dt}{t}.$$

Using (14), we estimate: $\mathbf{W}[(\mathbf{W}\sigma)^r d\sigma](x) \geq$

$$\geq C^{\frac{r}{p-1}} \int_0^\infty \Big[\int_t^\infty \left(\frac{\sigma(B(x,s))}{s^{n-\alpha p}} \right)^{\frac{1}{p-1}} \frac{ds}{s} \Big]^{\frac{r}{p-1}} \left(\frac{\sigma(B(x,t))}{t^{n-\alpha p}} \right)^{\frac{1}{p-1}} \frac{dt}{t}.$$

Proof of Lemma 4

Integrating by parts on the right-hand side, we deduce

$$\begin{aligned} &\mathbf{W}[(\mathbf{W}\sigma)^r d\sigma](x) \\ \geq &\frac{C^{\frac{r}{p-1}}}{\frac{r}{p-1}+1} \left(\int_0^\infty \left(\frac{\sigma(B(x,s))}{s^{n-\alpha p}} \right)^{\frac{1}{p-1}} \frac{ds}{s} \right)^{\frac{r}{p-1}+1} \\ = &\frac{C^{\frac{r}{p-1}}}{\frac{r}{p-1}+1} \left(\mathbf{W}_{1,p}\sigma(x) \right)^{\frac{r}{p-1}+1}. \end{aligned}$$

Since $\frac{r}{p-1}+1 \leq e^{\frac{r}{p-1}}$, we have

$$\frac{C^{\frac{r}{p-1}}}{\frac{r}{p-1}+1} \geq (C\, e^{-1})^{\frac{r}{p-1}},$$

which completes the proof of (13) with $\mathfrak{c} = C\, e^{-1}$. □

Proof of the lower bound (A)

To prove estimate (A), let $d\omega = u^q d\sigma$. Fix $x \in \mathbb{R}^n$ and pick $R > |x|$. Let $B = B(0, R)$, and $d\sigma_B = \chi_B \, d\sigma.$ Since u is a supersolution,

$$u(x) \geq \mathrm{W}\left[(\mathrm{W}\omega)^q d\sigma_B\right](x)$$
$$= \int_0^\infty \left(\frac{1}{t^{n-p}} \int_{B(x,t)\cap B} [\mathrm{W}\omega(z)]^q d\sigma(z)\right)^{\frac{1}{p-1}} \frac{dt}{t}.$$

Obviously, $\inf_{B(x,t)\cap B} [\mathrm{W}\omega] \geq \inf_B [\mathrm{W}\omega]$. By the Wolff potential lemma again, there exists $C = C(\alpha, p, n) > 0$ so that

$$\inf_B [\mathrm{W}\omega] \geq C \int_R^\infty \left(\frac{\omega(B(0,s))}{s^{n-p}}\right)^{\frac{1}{p-1}} \frac{ds}{s}.$$

Proof of the lower bound (A)

Combining the preceding estimates, we estimate

$$u(x) \geq (C\, M(R))^{\frac{q}{p-1}}\ \mathrm{W}\sigma_B(x),$$

where

$$M(R) := \int_R^\infty \left(\frac{\omega(B(0,s))}{s^{n-p}}\right)^{\frac{1}{p-1}} \frac{ds}{s} > 0.$$

We use this estimate in (3), and invoke Lemma 4 with $r = q$ and σ_B in place of σ. This yields

$$\begin{aligned} u(x) &\geq \mathrm{W}(u^q d\sigma_B)(x) \\ &\geq (C\, M(R))^{(\frac{q}{p-1})^2}\ \mathrm{W}\left[(\mathrm{W}\sigma_B)^q d\sigma_B\right](x) \\ &\geq \mathfrak{c}^{\frac{q}{p-1}}\ (C\, M(R))^{(\frac{q}{p-1})^2}\ \left[\mathrm{W}\sigma_B(x)\right]^{1+\frac{q}{p-1}}. \end{aligned}$$

Proof of the lower bound (A)

Iterating this procedure and using Lemma 4 with $r = q \sum_{k=0}^{j-1} \left(\frac{q}{p-1}\right)^k$ and σ_B in place of σ, we deduce by induction,

$$u(x) \geq \mathfrak{c}^{\sum_{k=1}^{j} k \left(\frac{q}{p-1}\right)^k} (C\, M(R))^{\left(\frac{q}{p-1}\right)^{j+1}} \, [\mathrm{W}\sigma_B(x)]^{\sum_{k=0}^{j} \left(\frac{q}{p-1}\right)^k},$$

for all $j = 2, 3, \ldots$. Since $0 < q < p - 1$, obviously

$$\sum_{k=1}^{\infty} k \left(\frac{q}{p-1}\right)^k < \infty.$$

Letting $j \to \infty$ in the preceding estimate we obtain

$$u(x) \geq C \, [\mathrm{W}\sigma_B(x)]^{\frac{p-1}{p-1-q}}, \quad B = B(0, R), \quad R > |x|,$$

where $C = C(\alpha, p, q, n)$. Letting $R \to \infty$ yields (A) for all $x \in \mathbb{R}^n$. $\square$

Proof of the lower bound (B)

We will need the following key lemma. Its proof is based on Vitali's covering lemma, and weak-type maximal function inequalities.

Lemma 5

Let $1 < p < \infty$, $0 < q < p-1$, $0 < \alpha < \frac{n}{p}$, *and* $\sigma \in \mathcal{M}^+(\mathbb{R}^n)$. *Suppose* $u \in L^q_{\mathrm{loc}}(\mathbb{R}^n, d\sigma)$ *is a nontrivial* **supersolution**. *Then there exists a constant* $C = C(\alpha, p, q, n)$ *so that, for every ball* B,

$$\varkappa(B) \le C \left(\int_B u^q d\sigma \right)^{\frac{p-1-q}{q(p-1)}}. \tag{15}$$

Remarks. 1. If $u \in L^q(\mathbb{R}^n, d\sigma)$ **globally** in Lemma 5, then clearly

$$\varkappa \le C \left(\int_{\mathbb{R}^n} u^q d\sigma \right)^{\frac{p-1-q}{q(p-1)}}. \tag{16}$$

2. An analogue of (16) was proved for **(QS)&(WMP)** kernels in Lect. 4.

Proof of the lower bound (B)

Proof of Lemma 5. Let $d\omega = u^q d\sigma \in \mathcal{M}^+(\mathbb{R}^n)$. For $\nu \in \mathcal{M}^+(\mathbb{R}^n)$, consider the centered maximal function

$$M_\omega \nu(y) = \sup_{\rho>0} \left[\frac{\nu(B(y, \frac{\rho}{5}))}{\omega(B(y, \rho))} \right], \qquad y \in \mathbb{R}^n, \tag{17}$$

where we follow the convention $\frac{0}{0} = 0$. Let

$$E_t = \{y \in \mathbb{R}^n : \; M_\omega \nu(y) > t\}, \quad t > 0.$$

Suppose $E_t \neq \emptyset$. Then, for every $y \in E_t$, there exists a ball $B(y, \rho_y)$ such that

$$\frac{\nu(B(y, \frac{\rho_y}{5}))}{\omega(B(y, \rho_y))} > t.$$

Thus $E_t \subset \bigcup_{y \in E_t} B(y, \frac{\rho_y}{5})$, and hence for any compact set $F \subset E_t$ there exists a $k \in \mathbb{N}$ such that

$$F \subset \bigcup_{j=1}^{k} B\Big(y_j, \frac{\rho_{y_j}}{5}\Big).$$

Proof of the lower bound (B)

Applying Vitali's covering lemma, we find disjoint balls $\left\{ B\left(y_{j_i}, \frac{\rho_{y_{j_i}}}{5}\right)\right\}_{i=1}^{m}$ such that

$$F \subset \bigcup_{i=1}^{m} B\left(y_{j_i}, \rho_{y_{j_i}}\right).$$

Consequently,

$$\omega(F) \le \sum_{i=1}^{m} \omega\left(B(y_{j_i}, \rho_{y_{j_i}})\right) \le \frac{1}{t} \sum_{i=1}^{m} \nu\left(B(y_{j_i}, \frac{\rho_{y_{j_i}}}{5})\right) \le \frac{1}{t} \nu(\mathbb{R}^n).$$

Therefore, taking the supremum over all compact sets $F \subset E_t$, we obtain the weak-type $(1, 1)$ maximal function inequality,

$$\sup_{t>0} t\, \omega(E_t) := \|M_\omega \nu\|_{L^{1,\infty}(\mathbb{R}^n, d\omega)} \le \|\nu\|. \tag{18}$$

Proof of the lower bound (B)

Clearly, for any $y \in \mathbb{R}^n$ such that $M_\omega \nu(y) < \infty$, we have

$$
\begin{aligned}
\mathrm{W}\nu(y) &= \int_0^\infty \left(\frac{\nu(B(y,s))}{s^{n-\alpha p}} \right)^{\frac{1}{p-1}} \frac{ds}{s} \\
&= 5^{\frac{n-\alpha p}{p-1}} \int_0^\infty \left(\frac{\nu(B(y,\frac{s}{5}))}{s^{n-\alpha p}} \right)^{\frac{1}{p-1}} \frac{ds}{s} \\
&= 5^{\frac{n-\alpha p}{p-1}} \int_0^\infty \left(\frac{\nu(B(y,\frac{s}{5}))}{\omega(B(y,s))} \cdot \frac{\omega(B(y,s))}{s^{n-\alpha p}} \right)^{\frac{1}{p-1}} \frac{ds}{s} \\
&\le 5^{\frac{n-\alpha p}{p-1}} \, (M_\omega \nu(y))^{\frac{1}{p-1}} \, \mathrm{W}\omega(y) \\
&\le 5^{\frac{n-\alpha p}{p-1}} \, (M_\omega \nu(y))^{\frac{1}{p-1}} \, u(y) \quad d\sigma\text{-a.e.}
\end{aligned}
$$

Note that if $\nu(B(y,\frac{s}{5})) > 0$ but $\omega(B(y,s)) = 0$ for some $s > 0$ then $M_\omega \nu(y) = \infty$. However, the set of such y has ω-measure zero by (18), as well as σ_B-measure zero, since $\inf_B u \ge \inf_B [\mathrm{W}\omega] > 0$ for any B.

Proof of the lower bound (B)

Hence, by the preceding estimate with $c = 5^{\frac{q(n-\alpha p)}{p-1}}$, for any ball B,

$$\|W\nu\|^q_{L^q(\mathbb{R}^n, d\sigma_B)} \le c \int_B (M_\omega \nu)^{\frac{q}{p-1}}\, u^q d\sigma = c \int_B (M_\omega \nu)^{\frac{q}{p-1}}\, d\omega.$$

To complete our estimates, we invoke the well-known inequality

$$\|f\|_{L^r(X,\omega)} \le c(r)\, \omega(X)^{\frac{1-r}{r}}\, \|f\|_{L^{1,\infty}(X,\omega)},$$

where $0 < r < 1$, and $\omega \in \mathcal{M}^+(X)$. Applying this inequality with $X = B$, $r = \frac{q}{p-1}$ and $f = M_\omega \nu$, together with (18), we estimate

$$\begin{aligned}\|W\nu\|^q_{L^q(\mathbb{R}^n, d\sigma_B)} &\le C\, \omega(B)^{1-\frac{q}{p-1}}\, \|M_\omega \nu\|^{\frac{q}{p-1}}_{L^{1,\infty}(\mathbb{R}^n, d\omega)} \\ &\le C\, \omega(B)^{1-\frac{q}{p-1}}\, \|\nu\|^{\frac{q}{p-1}},\end{aligned}$$

where $C = C(\alpha, p, q, n)$.

Proof of the lower bound (B)

To prove estimate (B), we observe that, for any supersolution $\boldsymbol{u}$,

$$u(x) \geq \mathrm{W}(u^q d\sigma) = \int_0^\infty \left[\frac{\int_{B(x,s)} u^q \, d\sigma}{s^{n-\alpha p}} \right]^{\frac{1}{p-1}} \frac{ds}{s}$$

By Lemma 5, for some $C = C(\alpha, p, q, n)$,

$$\int_{B(x,s)} u^q d\sigma \geq C \, [\varkappa(B(x,s))]^{\frac{q(p-1)}{p-1-q}}, \quad \forall x \in \mathbb{R}^n, \; s > 0.$$

Thus,

$$u(x) \geq C \int_0^\infty \left[\frac{\varkappa(B(x,s))^{\frac{q(p-1)}{p-1-q}}}{s^{n-\alpha p}} \right]^{\frac{1}{p-1}} \frac{ds}{s} = C \, \mathrm{K}\sigma(x).$$

This completes the proof of estimate (B), and hence Theorem 25. □

Corollary of Theorem 25

As a consequence of Theorem 25 and Lemma 3, we obtain the following corollary.

Corollary

Under the assumptions of Theorem 25, there exist constants $C_i = C_i(\alpha, p, q, n) > 0$ *(* $i = 1, 2$ *) so that*

$$\begin{aligned} C_1 \left[(\mathbf{W}\sigma(x))^{\frac{p-1}{p-1-q}} + \mathbf{K}\sigma(x) \right] &\le u(x) \\ &\le C_2 \left[(\mathbf{W}\sigma(x))^{\frac{p-1}{p-1-q}} + \mathbf{K}\sigma(x) \right], \qquad x \in \mathbb{R}^n, \end{aligned} \tag{19}$$

for any solution u *to* (1)*. These estimates also hold* $d\sigma$*-a.e. Moreover, the lower estimate holds for any* **supersolution** u *such that inequality* (3) *holds at* $x \in \mathbb{R}^n$*, whereas the upper estimate holds for any* **subsolution** u *such that inequality* (2) *holds at* $x \in \mathbb{R}^n$*.*

Some useful estimates of constants $\varkappa$

Let $1 < p < \infty$, $0 < \alpha < \frac{n}{p}$, and $0 < q < p-1$. Let $\sigma \in \mathcal{M}^+(\mathbb{R}^n)$. Recall that we denote by $\varkappa$ the least constant in the $(1, q)$-weighted norm inequality

$$\|\mathbf{W}\nu\|_{L^q(\mathbb{R}^n, d\sigma)} \leq \varkappa \, \|\nu\|^{\frac{1}{p-1}}, \quad \forall \nu \in \mathcal{M}^+(\mathbb{R}^n). \tag{20}$$

We will also need a localized version of (20) for $\sigma_E = \sigma|_E$, where E is a Borel subset of $\mathbb{R}^n$, and $\varkappa(E)$ is the least constant in the inequality

$$\|\mathbf{W}\nu\|_{L^q(\mathbb{R}^n, d\sigma_E)} \leq \varkappa(E) \, \|\nu\|^{\frac{1}{p-1}}, \quad \forall \nu \in \mathcal{M}^+(\mathbb{R}^n). \tag{21}$$

In applications, we often use $\varkappa(E)$ where $E = B$ is a ball in $\mathbb{R}^n$.

In the following lemma, we give lower and upper estimates of $\varkappa$ in terms of the norms of $\mathbf{W}\sigma$ in Lorentz spaces $L^{s,q}(\mathbb{R}^n, d\sigma)$ equipped with quasi-norm

$$\|f\|_{L^{s,q}(\mathbb{R}^n, d\sigma)} = \left(s \int_0^\infty t^q (\sigma\{x : |f(x)| \geq t\})^{\frac{q}{s}} \right)^{\frac{1}{q}}.$$

Some useful estimates of constants $\varkappa$

Lemma 6

Suppose $\sigma \in \mathcal{M}^+(\mathbb{R}^n)$, $1 < p < \infty$, and $0 < \alpha < \frac{n}{p}$. Then

$$C_1 \, \|\mathbf{W}\sigma\|_{L^{\frac{q(p-1)}{p-1-q}}(\mathbb{R}^n, d\sigma)} \le \varkappa \le C_2 \, \|\mathbf{W}\sigma\|_{L^{\frac{q(p-1)}{p-1-q}, q}(\mathbb{R}^n, d\sigma)}, \qquad (22)$$

where C_1, C_2 are positive constants which depend only on p, q, α, n.

Proof of Lemma 6. Clearly it suffices to consider the case $\sigma \neq 0$. To prove the lower estimate in (22), we may assume without loss of generality that $\varkappa < \infty$. Then by [Cao-V. 2017], Theorem 4.4, there exists a positive solution $u \in L^q(\mathbb{R}^n, d\sigma)$ to the equation $u = \mathbf{W}(u^q d\sigma)$. For $d\nu = u^q d\sigma$, we have $\mathbf{W}\nu = u$, and (20) yields

$$\varkappa \ge \|u\|_{L^q(\mathbb{R}^n, d\sigma)}^{\frac{p-1-q}{p-1}}.$$

Some useful estimates of constants $\varkappa$

On the other hand, by the lower estimate (A) above, there exists a constant $C = C(\alpha, p, q, n)$ such that

$$u(x) \geq C \, (\mathbf{W}\sigma(x))^{\frac{p-1}{p-1-q}}, \quad x \in \mathbb{R}^n.$$

Combining the preceding estimates gives the lower estimate in (22).

To prove the upper estimate in (22), without loss of generality we may assume that $\mathbf{W}\sigma < \infty$ $d\sigma$-a.e. Otherwise both sides of the upper estimate are infinite due to the lower estimate in (22).

Let $\nu \in \mathcal{M}^+(\mathbb{R}^n)$. By duality (Hölder's inequality) for Lorentz spaces,

$$\|\mathbf{W}\nu\|^q_{L^q(\mathbb{R}^n, d\sigma)} = \int_{\mathbb{R}^n} \left(\frac{\mathbf{W}\nu}{\mathbf{W}\sigma}\right)^q (\mathbf{W}\sigma)^q d\sigma$$

$$\leq c(q,p) \left\| \left(\frac{\mathbf{W}\nu}{\mathbf{W}\sigma}\right)^q \right\|_{L^{\frac{p-1}{q},\infty}(\mathbb{R}^n, d\sigma)} \|(\mathbf{W}\sigma)^q\|_{L^{\frac{p-1}{p-1-q},1}(\mathbb{R}^n, d\sigma)}$$

$$= c(q,p) \left\| \frac{\mathbf{W}\nu}{\mathbf{W}\sigma} \right\|^q_{L^{p-1,\infty}(\mathbb{R}^n, d\sigma)} \|\mathbf{W}\sigma\|^q_{L^{\frac{q(p-1)}{p-1-q},q}(\mathbb{R}^n, d\sigma)}.$$

Some useful estimates of constants $\varkappa$

For $\nu, \sigma \in \mathcal{M}^+(\mathbb{R}^n)$, we use again the maximal function

$$M_\sigma \nu(y) = \sup_{\rho>0} \left[\frac{\nu(B(y, \frac{\rho}{5}))}{\sigma(B(y, \rho))}\right], \qquad y \in \mathbb{R}^n.$$

As was verified above (Proof of Lemma 5), for $c = 5^{\frac{n-\alpha p}{p-1}}$,

$$\frac{\mathbf{W}\nu(y)}{\mathbf{W}\sigma(y)} \le c\,(M_\sigma \nu(y))^{\frac{1}{p-1}}, \qquad y \in \mathbb{R}^n, \tag{23}$$

and consequently by the weak $(1,1)$ maximal function inequality,

$$\left\|\frac{\mathbf{W}\nu}{\mathbf{W}\sigma}\right\|_{L^{p-1,\infty}(\mathbb{R}^n, d\sigma)} \le c \left\|(M_\sigma \nu)^{\frac{1}{p-1}}\right\|_{L^{p-1,\infty}(\mathbb{R}^n, d\sigma)}$$

$$= c\, \|M_\sigma \nu\|_{L^{1,\infty}(\mathbb{R}^n, d\sigma)}^{\frac{1}{p-1}} \le c\, \|\nu\|^{\frac{1}{p-1}}.$$

Some useful estimates of constants $\varkappa$

Combining the preceding estimates, we obtain

$$\|\mathbf{W}\nu\|_{L^q(\mathbb{R}^n, d\sigma)} \le C\, \|\mathbf{W}\sigma\|_{L^{\frac{q(p-1)}{p-1-q}, q}(\mathbb{R}^n, d\sigma)} \, \|\nu\|^{\frac{1}{p-1}},$$

which completes the proof of the upper estimate in (22). □

For $1 < p < n$ and $0 < \alpha < \frac{n}{p}$, the Riesz capacity of a measurable set $E \subset \mathbb{R}^n$ is defined by

$$\mathrm{cap}_{\alpha,p}(E) = \inf \left\{ \|g\|^p_{L^p(\mathbb{R}^n)} \colon \, \mathrm{I}_\alpha g(x) \ge 1 \text{ on } E, \;\; g \in L^p_+(\mathbb{R}^n) \right\}.$$

We will often prefer to use the simplified notation $\mathrm{cap}(\cdot) = \mathrm{cap}_{\alpha,p}(\cdot)$.

In the case $\alpha = 1$, it is known that $\mathrm{cap}_{1,p}(F) \approx \mathrm{cap}_p(F)$ for all compact sets $F \subset \mathbb{R}^n$, where $\mathrm{cap}_p(F)$ is the p-capacity, and the constants of equivalence depend only on p and n.

Some useful estimates of constants $\varkappa$

Definition. Let $\sigma \in \mathcal{M}^+(\mathbb{R}^n)$. Then σ is a Maz'ya measure if there exists a constant $\mathfrak{m} > 0$ such that

$$\sigma(F) \le \mathfrak{m}\, \mathrm{cap}(F), \quad \text{for all compact sets } F \subset \mathbb{R}^n \tag{24}$$

By the known properties of Riesz capacities, condition (24) actually holds for all Borel sets $E \subset \mathbb{R}^n$ in place of F.

Lemma 7

Suppose $1 < p < n$, $0 < \alpha < \frac{n}{p}$, $0 < q < p-1$, *and* $\sigma \in \mathcal{M}^+(\mathbb{R}^n)$.
(a) *If σ satisfies condition* (24), *then*

$$[\varkappa(E)]^{\frac{q(p-1)}{p-1-q}} \le C_3\, \sigma(E), \quad \text{for all Borel sets } E \subset \mathbb{R}^n, \tag{25}$$

where $C_3 = C_3(\alpha, p, q, \mathfrak{m}, n)$.

Proof of Lemma 7

Lemma 7 (continuation)

(b) *If σ satisfies condition* (24), *then*

$$\mathbf{K}\sigma(x) \le C_4 \mathbf{W}\sigma(x), \quad x \in \mathbb{R}^n, \tag{26}$$

where $C_4 = C_3^{\frac{1}{p-1}}$.

Proof of Lemma 7. It is known [Cao-V. 2017], Lemma 2.1, that if (24) holds, then for every $s > 0$,

$$\int_E (\mathbf{W}\sigma_E)^s d\sigma \le C_5\, \sigma(E), \quad \text{for all Borel sets } E \subset \mathbb{R}^n, \tag{27}$$

where $C_5 = C_5(\alpha, p, s, \mathfrak{m}, n)$. We will give a simpler proof [Verbitsky 2021] of (27) avoiding discrete Wolff potentials and random shifts of the dyadic lattice used in [Cao-V. 2017].

Proof of Lemma 7

We start with the well-known trace inequality for Riesz potentials [Maz'ya 2011], there exists a constant $C_6 = C_6(\alpha, p, n)$ so that

$$||\mathbf{I}_\alpha f||_{L^p(\mathbb{R}^n, d\sigma)} \le C_6 \, m^{\frac{1}{p}} \, ||f||_{L^p(\mathbb{R}^n, dx)}, \quad \forall f \in L^p(\mathbb{R}^n, dx),$$

where $1 < p < \infty$, which is equivalent to condition (24).
We rewrite this inequality in the equivalent dual form,

$$||\mathbf{I}_\alpha (g d\sigma)||_{L^{p'}(\mathbb{R}^n, dx)} \le C_6 \, m^{\frac{1}{p}} \, ||g||_{L^{p'}(\mathbb{R}^n, d\sigma)}, \quad \forall g \in L^{p'}(\mathbb{R}^n, d\sigma).$$

Let $g \ge 0$, and $d\omega = g d\sigma$. By Wolff's inequality (see Lecture 5),

$$||\mathbf{I}_\alpha \omega||^{p'}_{L^{p'}(\mathbb{R}^n, dx)} \ge C_7 \int_{\mathbb{R}^n} \mathbf{W}\omega \, d\omega,$$

where $C_7 = C_7(\alpha, p, n)$.

Proof of Lemma 7

Hence, for some $C_8 = C_8(\alpha, p, \mathfrak{m}, n)$,

$$\int_{\mathbb{R}^n} \mathbf{W}(g d\sigma)\, g\, d\sigma \leq C_8 \, \|g\|^{p'}_{L^{p'}(\mathbb{R}^n, d\sigma)}, \quad \forall g \in L^{p'}(\mathbb{R}^n, d\sigma). \tag{28}$$

Letting $g = \chi_E$ in (28) gives

$$\int_{\mathbb{R}^n} \mathbf{W}\sigma_E \, d\sigma_E \leq C_8 \, \sigma(E). \tag{29}$$

Also, letting $g = \chi_E (\mathbf{W}\sigma_E)^r$ with $r > 0$ in (28) yields

$$\int_{\mathbb{R}^n} \mathbf{W}[(\mathbf{W}\sigma_E)^r d\sigma_E] \, (\mathbf{W}\sigma_E)^r d\sigma_E \leq C_8 \int_{\mathbb{R}^n} (\mathbf{W}\sigma_E)^{rp'} d\sigma_E.$$

Proof of Lemma 7

Applying Lemma 4 to the measure σ_E, we obtain

$$\mathbf{W}[(\mathbf{W}\sigma_E)^r d\sigma_E] \geq \mathfrak{c}^{\frac{r}{p-1}} (\mathbf{W}\sigma_E)^{\frac{r}{p-1}+1},$$

where $\mathfrak{c} = \mathfrak{c}(\alpha, p, n)$. Combining this estimate with the preceding inequality gives

$$\int_{\mathbb{R}^n} (\mathbf{W}\sigma_E)^{rp'+1} d\sigma_E \leq C_9 \int_{\mathbb{R}^n} (\mathbf{W}\sigma_E)^{rp'} d\sigma_E,$$

where $C_9 = C_9(\alpha, p, r, \mathfrak{m}, n)$. Letting $r = \frac{j}{p'}$, for all $j \in \mathbb{N}$ we deduce

$$\int_{\mathbb{R}^n} (\mathbf{W}\sigma_E)^{j+1} d\sigma_E \leq C_{10} \int_{\mathbb{R}^n} (\mathbf{W}\sigma_E)^{j} d\sigma_E,$$

where $C_{10} = C_{10}(\alpha, p, \mathfrak{m}, j, n)$.

Proof of Lemma 7

By (29), the preceding inequality holds for $j = 0$. Hence by induction,

$$\int_{\mathbb{R}^n} (W\sigma_E)^j \, d\sigma_E \le C_{11}\, \sigma(E), \quad j = 0, 1, \ldots .$$

where $C_{11} = C_{11}(\alpha, p, \mathfrak{m}, j, n)$. This proves (27) for $s = j$. The general case $j \le s < j + 1$ follows using Hölder's inequality. This completes the proof of (27) for all $s > 0$ with the constant $C_5 = C_5(\alpha, p, \mathfrak{m}, s, n)$.

We are now ready to complete the proof of Lemma 7. By Lemma 6, using the upper estimate in (22) for σ_E in place of σ, we obtain

$$\varkappa(E) \le C_2 \, \|W\sigma_E\|_{L^{\frac{q(p-1)}{p-1-q}, q}(\mathbb{R}^n, d\sigma_E)} . \tag{30}$$

We next invoke the known inequality for Lorentz spaces,

$$\|f\|_{L^{s_1, q}(\mathbb{R}^n, d\sigma_E)} \le C(s_1, s) \, [\sigma(E)]^{\frac{1}{s_1} - \frac{1}{s}} \, \|f\|_{L^s(\mathbb{R}^n, d\sigma_E)} ,$$

if $s > s_1$, for any $q > 0$.

Proof of Lemma 7

Applying this estimate with $s_1 = \frac{q(p-1)}{p-1-q}$ and any $s > \frac{q(p-1)}{p-1-q}$ gives

$$\|\mathbf{W}\sigma_E\|_{L^{\frac{q(p-1)}{p-1-q},q}(\mathbb{R}^n, d\sigma_E)} \le C\,[\sigma(E)]^{\frac{p-1-q}{q(p-1)}-\frac{1}{s}}\,\|\mathbf{W}\sigma_E\|_{L^s(\mathbb{R}^n, d\sigma_E)}.$$

Inequality (27) now yields

$$\|\mathbf{W}\sigma_E\|_{L^{\frac{q(p-1)}{p-1-q},q}(\mathbb{R}^n, d\sigma_E)} \le C\,[\sigma(E)]^{\frac{p-1-q}{q(p-1)}}.$$

Combining this estimate with (30) yields

$$\varkappa(E) \le C\,[\sigma(E)]^{\frac{p-1-q}{q(p-1)}},$$

where $C = C(\alpha, p, q, \mathfrak{m}, n)$. This completes the proof of (25), that is, statement (a).

Proof of Lemma 7

To prove statement (b), it suffices to apply statement (a) in the special case $E = B(x, r)$, which gives

$$[\varkappa(B(x,r))]^{\frac{q(p-1)}{p-1-q}} \le C\,\sigma(B(x,r)),$$

where $C = C(\alpha, p, q, \mathfrak{m}, n)$. Hence, by the definition of the intrinsic potential $\mathbf{K}$, we immediately have

$$\begin{aligned} \mathbf{K}\sigma(x) &= \int_0^\infty \left[\frac{\varkappa(B(x,s))^{\frac{q(p-1)}{p-1-q}}}{s^{n-\alpha p}}\right]^{\frac{1}{p-1}} \frac{ds}{s} \\ &\le C^{\frac{1}{p-1}} \int_0^\infty \left[\frac{\sigma(B(x,s))}{s^{n-\alpha p}}\right]^{\frac{1}{p-1}} \frac{ds}{s} = C^{\frac{1}{p-1}}\, \mathbf{W}\sigma(x). \end{aligned}$$

This completes the proof of statement (b), and hence Lemma 7. □

Remarks on Brezis–Kamin type estimates

Remarks. 1. Lemma 7 demonstrates that under assumption (24), the intrinsic potential $\mathbf{K}\sigma$ can be replaced with $\mathbf{W}\sigma$ in the upper pointwise estimate of any nontrivial subsolution u:

$$u(x) \le C \left[(\mathbf{W}\sigma(x))^{\frac{p-1}{p-1-q}} + \mathbf{W}\sigma(x) \right], \quad x \in \mathbb{R}^n.$$

2. In the special case $\alpha = 1$, Lemma 7 shows that, for Maz'ya measures such that

$$\sigma(F) \le \mathfrak{m}\, \mathrm{cap}_p(F), \quad \text{for all compact sets } F \subset \mathbb{R}^n,$$

actually Theorem 21 (Brezis–Kamin type estimates) is an immediate consequence of the general pointwise estimates of Theorem 22. Moreover, for such measures a solution to the homogeneous problem

$$-\mathrm{div}\mathcal{A}(x, \nabla u) = \sigma u^q \quad \text{in } \mathbb{R}^n, \qquad \liminf_{x\to\infty} u(x) = 0,$$

in the case $0 < q < p-1$ exists if and only if $\mathbf{W}_{1,p}\sigma \not\equiv \infty$.

Non-homogeneous integral equations

We next deduce estimates for sub- and super-solutions to the equation

$$u = \mathbf{W}(u^q d\sigma) + \mathbf{W}\mu, \quad u \ge 0 \quad \text{in } \mathbb{R}^n, \tag{31}$$

in the case $0 < q < p - 1$. We assume here that $\mu \neq 0$. In particluar, all solutions u to (31) are nontrivial: $u \ge \mathbf{W}\mu > 0$, and $u < \infty$ $d\sigma$-a.e.

Theorem 26 (Verbitsky 2021)

Let $1 < p < \infty$, $0 < \alpha < \frac{n}{p}$, $0 < q < p - 1$. Let $\sigma, \mu \in \mathcal{M}^+(\mathbb{R}^n)$ ($\mu \neq 0$). Then there exist positive constants C_1, C_2 which depend only on p, q, α and n such that any nonnegative solution u to (31) satisfies the estimates

$$\begin{aligned} C_1 \left[(\mathbf{W}\sigma(x))^{\frac{p-1}{p-1-q}} + \mathbf{K}\sigma(x) + \mathbf{W}\mu(x) \right] &\le u(x) \\ \le C_2 \left[(\mathbf{W}\sigma(x))^{\frac{p-1}{p-1-q}} + \mathbf{K}\sigma(x) + \mathbf{W}\mu(x) \right], &\qquad x \in \mathbb{R}^n. \end{aligned} \tag{32}$$

Non-homogeneous integral equations

Theorem 26 (continuation)

The upper estimate in (32) *holds at every* $\boldsymbol{x}$ *where* $\boldsymbol{u(x) < \infty}$, *and consequently* $\boldsymbol{d\sigma}$*-a.e.*
Moreover, the lower estimate in (32) *holds for every* **supersolution** $\boldsymbol{u}$ *at every* $\boldsymbol{x} \in \mathbb{R}^{\boldsymbol{n}}$, *that is, if*

$$\mathbf{W}(u^q d\sigma)(x) + \mathbf{W}\mu(x) \leq u(x) < \infty \quad d\sigma\text{-}a.e., \tag{33}$$

whereas the upper estimate holds for every **subsolution** $\boldsymbol{u}$, *both* $\boldsymbol{d\sigma}$*-a.e., and at every* $\boldsymbol{x} \in \mathbb{R}^{\boldsymbol{n}}$ *such that*

$$u(x) \leq \mathbf{W}(u^q d\sigma)(x) + \mathbf{W}\mu(x) < \infty. \tag{34}$$

Non-homogeneous integral equations

Proof. Since $\mu \neq 0$, we have

$$u(x) \geq \mathbf{W}\mu(x) > 0, \quad \forall x \in \mathbb{R}^n.$$

Clearly, any supersolution of equation (31) is also a supersolution of the homogeneous equation (1). Hence, by the Corollary of Theorem 25, there exists a positive constant $c = c(p, q, \alpha, n)$ such that

$$u(x) \geq c \left[(\mathbf{W}\sigma(x))^{\frac{p-1}{p-1-q}} + \mathbf{K}\sigma(x) \right], \quad \forall x \in \mathbb{R}^n.$$

These two lower estimates combined yield the lower bound in (32) with $C_1 = C_1(p, q, \alpha, n) > 0$.

To prove the upper bound, for any subsolution u to (31), we fix $x \in \mathbb{R}^n$ such that $u(x) \leq \mathbf{W}(u^q d\sigma)(x) + \mathbf{W}\mu(x) < \infty$. Notice that if u is a solution to (31), then this is equivalent to $u(x) < \infty$.

Non-homogeneous integral equations

Let $d\omega = u^q d\sigma + d\mu$, $c_1 = \max(1, 2^{\frac{p-2}{p-1}})$ and $c_2 = \max(1, 2^{\frac{1}{2-p}})$. We obviously have $u(x) \le c_1 \, \mathbf{W}\omega(x) < \infty$ at x and $d\sigma$-a.e. It follows,

$$\begin{aligned} \mathbf{W}\omega(x) &= \mathbf{W}(u^q d\sigma + d\mu)(x) \\ &\le c_2 \, \mathbf{W}(u^q d\sigma)(x) + c_2 \, \mathbf{W}\mu(x) \\ &\le c_1^q \, c_2 \, \mathbf{W}[(\mathbf{W}\omega)^q d\sigma](x) + c_2 \, \mathbf{W}\mu(x). \end{aligned}$$

By Lemma 2 with ω in place of ν, we have for some $C = C(\alpha, p, q, n)$,

$$\mathbf{W}[(\mathbf{W}\omega)^q d\sigma](x) \le C \, (\mathbf{W}\omega(x))^{\frac{q}{p-1}} \left[(\mathbf{W}\sigma(x))^{\frac{p-1}{p-1-q}} + \mathbf{K}\sigma(x) \right]^{\frac{p-1-q}{p-1}} .$$

Combining the preceding estimates we deduce

$$\begin{aligned} \mathbf{W}\omega(x) \le c_1^q \, c_2 \, C \, (\mathbf{W}\omega(x))^{\frac{q}{p-1}} \left[(\mathbf{W}\sigma(x))^{\frac{p-1}{p-1-q}} + \mathbf{K}\sigma(x) \right]^{\frac{p-1-q}{p-1}} \\ + c_2 \, \mathbf{W}\mu(x). \end{aligned}$$

Non-homogeneous integral equations

Using Young's inequality with exponents $\frac{p-1}{q}$ and $\frac{p-1}{p-1-q}$ in the first term on the right-hand side, we estimate

$$\mathbf{W}\omega(x) \le \frac{1}{2}\mathbf{W}\omega(x) + C_1 \left[(\mathbf{W}\sigma(x))^{\frac{p-1}{p-1-q}} + \mathbf{K}\sigma(x) \right] + c_2\, \mathbf{W}\mu(x),$$

where $C_1 = C_1(\alpha, p, q, n)$ is a positive constant.

Since $\mathbf{W}\omega(x) < \infty$, we can move the first term on the right to the left-hand side, and obtain

$$u(x) \le c_1\, \mathbf{W}\omega(x) \le C_2 \left[(\mathbf{W}\sigma(x))^{\frac{p-1}{p-1-q}} + \mathbf{K}\sigma(x) + \mathbf{W}\mu(x) \right],$$

where $C_2 = C_2(\alpha, p, q, n)$ is a positive constant. This completes the proof of the upper estimate in (32) and Theorem 26. □

Bilateral pointwise estimates for quasi-linear equations

We now give a proof of bilateral pointwise estimates

$$u(x) \approx (\mathbf{W}_{1,p}\sigma(x))^{\frac{p-1}{p-1-q}} + \mathbf{K}_{1,p}\sigma(x) + \mathbf{W}_{1,p}\mu(x), \qquad (35)$$

for all nontrivial $\mathcal{A}$-superharmonic solutions of the quasi-linear equation

$$-\operatorname{div}\mathcal{A}(x, \nabla u) = \sigma u^q + \mu \quad \text{in } \mathbb{R}^n, \qquad \liminf_{x\to\infty} u(x) = 0, \qquad (36)$$

in the case $0 < q < p-1$, where $\mu, \sigma \in \mathcal{M}^+(\mathbb{R}^n)$. The constants of equivalence in (35) depend only on p, q, n.

Remark. A proof of the lower estimate for **all** such solutions, along with the upper estimate in (35) in the case $\mu = 0$ for the minimal solution was provided in [Cao-Verbitsky 2017].

Bilateral pointwise estimates

Theorem 27 (Verbitsky 2021)

Let $1 < p < n$ and $0 < q < p-1$. Let $\mu, \sigma \in \mathcal{M}^+(\mathbb{R}^n)$. Then there exists a nontrivial (super) solution u to (36) such that $\liminf_{|x|\to+\infty} u(x) = 0$ if and only if the following conditions hold:

$$\int_1^\infty \left(\frac{\mu(B(0,r))}{r^{n-p}}\right)^{\frac{1}{p-1}} \frac{dr}{r} + \int_1^\infty \left(\frac{\sigma(B(0,r))}{r^{n-p}}\right)^{\frac{1}{p-1}} \frac{dr}{r} < \infty, \quad (37)$$

$$\int_1^\infty \frac{(\varkappa(B(0,r)))^{\frac{q(p-1)}{p-1-q}}}{r^{n-p}} \frac{dr}{r} < \infty. \quad (38)$$

Under conditions (37), (38) any nontrivial $\mathcal{A}$-superharmonic solution u satisfies global estimates (35).
Moreover, the lower bound in (35) holds for every nontrivial **supersolution** u*, whereas the upper bound holds for every nontrivial* **subsolution** u.

Bilateral pointwise estimates

Proof. Let $d\omega = u^q d\sigma + d\mu$. If u is a solution to (36), then by the Kilpeläinen–Malý theorem on $\mathbb{R}^n$,

$$\mathfrak{C}^{-1} \mathrm{W}_{1,p}\omega(x) \le u(x) \le \mathfrak{C}\, \mathrm{W}_{1,p}\omega(x), \tag{39}$$

where $\mathfrak{C} = \mathfrak{C}(n, p)$ is a positive constant.
It is easy to see that the lower bound in (39) holds for $\mathcal{A}$-superharmonic $u \ge 0$ which are supersolutions, and the upper bound for subsolutions to

$$-\,\mathrm{div}\mathcal{A}(x, \nabla u) = \omega \quad \text{in } \mathbb{R}^n, \qquad \liminf_{x\to\infty} u(x) = 0, \tag{40}$$

Hence, for a nontrivial **supersolution** u to (36), we have

$$u \ge \mathfrak{C}^{-1} \mathrm{W}_{1,p}\omega \ge \mathrm{W}_{1,p}(u^q d\tilde{\sigma}) + \mathrm{W}_{1,p}\tilde{\mu},$$

with $\tilde{\mu} = c_1\mu$ and $\tilde{\sigma} = c_2\sigma$, if $c_i = c_i(p, q, n)$ are small enough.
Thus, the lower estimate (35) of Theorem 27 for supersolutions u follows from the lower estimate (32) of Theorem 26 in the special case $\alpha = 1$.

Bilateral pointwise estimates

Moreover, if a nontrivial (super) solution u to equation (36) exists, then by the just proved lower estimate (35) of Theorem 27 it follows that both $\mathbf{W}_{1,p}\mu \not\equiv \infty$ and $\mathbf{W}_{1,p}\sigma \not\equiv \infty$, and also $\mathbf{K}_{1,p}\sigma \not\equiv \infty$, which are equivalent to conditions (37) and (38) respectively.

Similarly, if u is a nontrivial subsolution to (36), then we need to pick the constants $c_i = c_i(p, q, C)$, $i = 1, 2$, large enough, so that, for scaled $\tilde{\mu} = c_1\mu$ and $\tilde{\sigma} = c_2\sigma$, we have

$$u \le \mathbf{W}_{1,p}(u^q d\tilde{\sigma}) + \mathbf{W}_{1,p}\tilde{\mu},$$

applying (39) for $d\omega = u^q d\sigma + d\mu$ again. Then the upper estimate in (35) is deduced from the upper estimate in (32).

It remains to demonstrate that, subject to conditions (37), (38), there exists a nontrivial solution u to (36). We sketch a proof next.

Bilateral pointwise estimates

In the homogeneous case $\mu = 0$ $(\sigma \neq 0)$, a positive $\mathcal{A}$-superharmonic solution $u \in L^q_{\rm loc}(\mathbb{R}^n, \sigma)$ was constructed in [Cao-V. 2017], Theorem 1.1, by iterations using a sequence u_j of $\mathcal{A}$-superharmonic functions so that

$$- \operatorname{div} \mathcal{A}(x, \nabla u_{j+1}) = \sigma u_j^q \quad \text{in } \mathbb{R}^n, \quad j = 1, 2, \ldots, \tag{41}$$

and

$$u_j(x) \le C\, v(x), \quad x \in \mathbb{R}^n,$$

where $C = C(p, q, n)$ and v is a nontrivial solution to the integral equation

$$v = \mathbf{W}_{1,p}(v^q d\sigma) \quad \text{in } \mathbb{R}^n.$$

Then $\liminf_{x \to \infty} v(x) = 0$, and by the Corollary of Theorem 25,

$$v(x) \approx (\mathbf{W}_{1,p}\sigma(x))^{\frac{p-1}{p-1-q}} + \mathbf{K}_{1,p}\sigma(x) \quad \text{in } \mathbb{R}^n.$$

Bilateral pointwise estimates

It is important to choose the initial iteration u_1 properly, as an $\mathcal{A}$-superharmonic solution to the equation

$$-\operatorname{div}\mathcal{A}(x, \nabla u_1) = \omega \quad \text{in } \mathbb{R}^n, \qquad \liminf_{x\to\infty} u_1(x) = 0, \qquad (42)$$

where, for some positive constants $c_0 = c_0(p, q, n)$, $C = C(p, q, n)$,

$$d\omega = c_0\, v_0^q\, d\sigma, \quad v_0 = (\mathbf{W}_{1,p}\sigma(x))^{\frac{p-1}{p-1-q}} \le C\, v.$$

Notice that $d\omega \le c_0\, C^q\, v^q\, d\sigma$. Hence, for some constants $C_i = C_i(p, q, n) > 0$,

$$u_1(x) \le C_1\, \mathbf{W}_{1,p}\omega(x) \le c_0^{\frac{1}{p-1}}\, C_2\, \mathbf{W}_{1,p}(v^q\, d\sigma)$$
$$= c_0^{\frac{1}{p-1}}\, C_2\, v(x) \le v(x),$$

provided $c_0^{\frac{1}{p-1}}\, C_2 \le 1$. By induction, we verify that, for small c_0,

$$u_j(x) \le u_{j+1}(x) \le v(x), \quad x \in \mathbb{R}^n, \quad j = 1, 2, \ldots.$$

See details in [Cao-V. 2017], proof of Theorem 1.1.

Bilateral pointwise estimates

Since $u_j \uparrow u \leq v$, it is not difficult to see using a Harnack type theorem that $u = \lim_{j\to\infty}$ is a nontrivial $\mathcal{A}$-superharmonic function, and

$$-\mathrm{div}\mathcal{A}(x, \nabla u_{j+1}) \longrightarrow -\mathrm{div}\mathcal{A}(x, \nabla u)$$

in the sense of measures [Trudinger-Wang 2002].
Passing to the limit in (41), we conclude that u is a nontrivial solution,

$$u(x) \leq v(x) \leq C \left[(\mathbf{W}_{1,p}\sigma(x))^{\frac{p-1}{p-1-q}} + \mathbf{K}_{1,p}\sigma(x) \right], \quad x \in \mathbb{R}^n.$$

In the case $\mu \neq 0$, a similar iteration argument can be used with $u_1 = 0$. Using [Phuc-V. 2009], Lemma 3.7 and Lemma 3.9, we can construct a nondecreasing sequence $u_j \uparrow u$ of $\mathcal{A}$-superharmonic functions so that

$$-\mathrm{div}\mathcal{A}(x, \nabla u_{j+1}) = \sigma u_j^q + \mu \quad \text{in } \mathbb{R}^n, \quad j = 1, 2, \ldots . \tag{43}$$

Bilateral pointwise estimates

This part of the construction actually works for any $q > 0$ and $p > 1$ (see the proof of Theorem 3.10 in [Phuc-V. 2009] for $q > p - 1$). However, for $0 < q < p - 1$ we control the growth of u_j differently. By the Kilpeläinen–Malý theorem on $\mathbb{R}^n$ (39), we have

$$\begin{aligned} u_{j+1} &\le \mathfrak{C}\, \mathbf{W}_{1,p}(\sigma u_j^q + \mu) \\ &\le \mathfrak{C} \max(1, 2^{\frac{2-p}{p-1}}) \left[\mathbf{W}_{1,p}(\sigma u_{j+1}^q) + \mathbf{W}_{1,p}\mu \right]. \end{aligned} \tag{44}$$

After scaling by letting $\tilde{\mu} = c^{p-1}\mu$ and $\tilde{\sigma} = c^{p-1}\sigma$, where the constant $c = \mathfrak{C} \max(1, 2^{\frac{2-p}{p-1}})$, we see that u_{j+1} is a subsolution for the corresponding integral equation (31), i.e.,

$$u_{j+1} \le \mathbf{W}_{1,p}(\tilde{\sigma} u_{j+1}^q) + \mathbf{W}_{1,p}\tilde{\mu}, \quad j = 0, 1, 2, \ldots. \tag{45}$$

Bilateral pointwise estimates

It follows by induction using Lemma 3 and the Corollary to Theorem 25 (with $\tilde{\mu}$ and $\tilde{\sigma}$) that the right-hand side of (45) is finite $d\sigma$-a.e. See details in [Verbitsky 2021].

For subsolutions u_{j+1}, we have the upper bound

$$u_{j+1}(x) \le C \left[(\mathrm{W}_{1,p}\sigma(x))^{\frac{p-1}{p-1-q}} + \mathrm{K}_{1,p}\sigma(x) + \mathrm{W}_{1,p}\mu(x) \right], \quad x \in \mathbb{R}^n,$$

with $C = C(p, q, n)$, where we switched back from $\tilde{\mu}$, $\tilde{\sigma}$ to μ, σ.

Passing again to the limit in (43), we deduce that $u = \lim_{j\to\infty} u_j$ is a nontrivial $\mathcal{A}$-superharmonic solution to (36), which satisfies the estimate

$$u(x) \le C \left[(\mathrm{W}_{1,p}\sigma(x))^{\frac{p-1}{p-1-q}} + \mathrm{K}_{1,p}\sigma(x) + \mathrm{W}_{1,p}\mu(x) \right] \qquad (46)$$

$d\sigma$-a.e., and at every $x \in \mathbb{R}^n$ where the right-hand side is finite.

□

Bilateral pointwise estimates

Remarks. 1. One of the technical difficulties in the construction of an $\mathcal{A}$-superharmonic solution to the equation

$$-\mathrm{div}\mathcal{A}(x, \nabla u) = \sigma\, u^q + \mu,$$

in the non-homogeneous case ($\mu \neq 0$), is that $\mu \in \mathcal{M}^+(\mathbb{R}^n)$ may be **singular** with respect to the p-capacity.

For such measures in general, the uniqueness problem and standard comparison principles for $\mathcal{A}$ superharmonic solutions to the equation

$$-\mathrm{div}\mathcal{A}(x, \nabla u) = \mu$$

are open in general. The iteration scheme described in [Phuc-Verbitsky 2008/09] relies instead on a **restricted version** of the comparison principle for a specifically constructed sequence of local renormalized solutions.

2. For solutions to the homogeneous equation ($\mu = 0$), $\sigma \in \mathcal{M}^+(\mathbb{R}^n)$ is indeed **absolutely continuous** with respect to the p-capacity, which simplifies the construction of iterations.