# Potential Theory and Nonlinear Elliptic Equations
# Lecture 4

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# Publications

1. A. Grigor'yan and I. Verbitsky, *Pointwise estimates of solutions to nonlinear equations for non-local operators*, ***Ann. Scuola Norm. Super. Pisa***, **20** (2020) 721–750.
2. A. Seesanea and I. Verbitsky, *Finite energy solutions to inhomogeneous nonlinear elliptic equations with sub-natural growth terms*, ***Adv. Calc. Var.***, **13** (2020) 53–74.
3. S. Quinn and I. Verbitsky, *A sublinear version of Schur's lemma and elliptic PDE*, ***Analysis & PDE***, **11** (2018) 439–466.
4. Dat Tien Cao and I. Verbitsky, *Nonlinear elliptic equations and intrinsic potentials of Wolff type*, ***J. Funct. Analysis***, **272** (2017) 112–165.
5. Dat Tien Cao and I. Verbitsky, *Pointwise estimates of Brezis–Kamin type for solutions of sublinear elliptic equations*, ***Nonlin. Analysis***, **146** (2016) 1–19.

# Additional literature


1. D. R. Adams and L. I. Hedberg, *Function Spaces and Potential Theory*, Grundlehren der math. Wissenschaften, **314**, Springer, Berlin, 1996.
2. A. Grigor'yan, *Heat Kernel and Analysis on Manifolds*, Amer. Math.Soc./Intern. Press Studies in Adv. Math., **47**, 2009.
3. N. S. Landkof, *Foundations of Modern Potential Theory*, Grundlehren der math. Wissenschaften, **180**, Springer, New York–Heidelberg, 1972.
4. V. G. Maz'ya, *Sobolev Spaces, with Applications to Elliptic Partial Differential Equations*, 2nd revised augm. ed., Grundlehren der math. Wissenschaften, **342**, Springer, Berlin, 2011.

# Integral inequalities for nondecreasing nonlinearities

**Theorem 10 (lower estimate)**

*Let $\sigma \in \mathcal{M}^+(\Omega)$, and let $K$ be a $\mathbf{(WMP)}$-kernel on $\Omega$ with constant $\mathfrak{b} \geq 1$. Let $g \colon [1, +\infty) \to [1, +\infty)$ be nondecreasing, continuous. If $\mathcal{A}u = K(g(u)d\sigma)$, and $u \geq \mathcal{A}u + 1$ $d\sigma$-a.e., then*

$$u(x) \geq 1 + \mathfrak{b}\left[F^{-1}\left(\mathfrak{b}^{-1}K\sigma(x)\right) - 1\right], \tag{1}$$

*for all $x \in \Omega$ such that $\mathcal{A}u(x) + 1 \leq u(x) < +\infty$, where necessarily*

$$\mathfrak{b}^{-1}K\sigma(x) < \ a := \int_1^{+\infty} \frac{ds}{g(s)}. \tag{2}$$

**Remarks.** 1. We will give below a proof of Theorem 10. A similar proof of Theorem 11 for noninreasing $g$ is omitted.
2. Theorem 9 with $g(t) = t^q$, but with any $h > 0$ in place of $1$ will be proved after that.

# Proof of Theorem 10

For any $t \geq 0$, we set as above,

$$\phi(t) = g(t+1) \quad \text{and} \quad \psi(t) = \phi(\mathfrak{b}^{-1}t) = g(\mathfrak{b}^{-1}t + 1). \tag{3}$$

As in the iterations lemma, define the sequence $\{f_k\}_{k=0}^{\infty}$ on $\Omega$ by

$$f_0 := K\sigma, \qquad f_{k+1} := K\left[(\phi(f_k))\, d\sigma\right].$$

We claim that, for all $k \geq 0$,

$$u \geq f_k + 1 \ \text{ in } \Omega. \tag{4}$$

Indeed, since $u \geq 1$, we have $u \geq \mathcal{A}1 + 1 = K\sigma + 1$, and consequently

$$u \geq \mathcal{A}u + 1 \geq f_0 + 1,$$

that is, (4) holds for $k = 0$. If (4) is already proved for some $k \geq 0$,

$$u \geq \mathcal{A}u + 1 \geq K\left[(\phi(f_k))\, d\sigma\right] + 1 = f_{k+1} + 1,$$

which completes the proof of (4).

# Proof of Theorem 10

(continuation)

Consider now the sequence $\{\psi_k\}_{k=0}^{\infty}$ on $[0,\infty)$ so that $\psi_0(t) := t$ and

$$\psi_{k+1}(t) := \int_0^t \psi \circ \psi_k(s) ds. \tag{5}$$

By the iterations lemma, we have, for all $x \in \Omega$ and $k \geq 0$,

$$f_k(x) \geq \psi_k(f_0(x)),$$

which together with (4) yield

$$u(x) \geq \psi_k(K\sigma(x)) + 1 \quad \text{for all } x \in \Omega.$$

By (3), the function $\psi$ is non-decreasing and $\psi \geq 1$, which implies that $\psi_{k+1}(t) \geq \psi_k(t)$ for all $t \geq 0$. Indeed, for $k = 0$ it follows from

$$\psi_1(t) = \int_0^t \psi(t)\, dt \geq t = \psi_0(t),$$

and $\psi_k \geq \psi_{k-1} \Longrightarrow \psi_{k+1} \geq \psi_k$ by (5) and the monotonicity of $\psi$.

# Proof of Theorem 10

(continuation)

We now set

$$\psi_\infty(t) := \lim_{k\to\infty} \psi_k(t) \,.$$

Hence, letting $k \to \infty$ in the preceding estimates, we deduce

$$u(x) \geq \psi_\infty(K\sigma(x)) + 1 \quad \text{for all } x \in \Omega. \tag{6}$$

Let us fix $x \in \Omega$ such that $u(x) < +\infty$. It follows from (6) that

$$t_0 := K\sigma(x) < +\infty \quad \text{and} \quad \psi_\infty(t_0) < \infty.$$

Without loss of generality we may assume that $t_0 > 0$ since in the case $K\sigma(x) = 0$ the desired estimates are obvious. We see that the function $\psi_\infty$ is finite on $[0, t_0]$, positive on $(0, t_0]$, and by the monotone convergence theorem, satisfies the integral equation

$$\psi_\infty(t) = \int_0^t \psi \circ \psi_\infty(s)\, ds, \quad 0 \leq t \leq t_0. \tag{7}$$

# Proof of Theorem 10

(continuation)

Hence, $\psi_\infty$ is continuously differentiable on $[0, t_0]$ and satisfies the ODE

$$\frac{d\psi_\infty}{dt} = \psi(\psi_\infty(t)), \quad \psi_\infty(0) = 0. \tag{8}$$

Setting

$$\Psi(\xi) = \int_0^\xi \frac{ds}{\psi(s)} = \mathfrak{b}\, F(1 + \mathfrak{b}^{-1}\xi) \tag{9}$$

and observing that by the Chain Rule and (8),

$$\frac{d\Psi(\psi_\infty)(t)}{dt} = \left(\frac{d\Psi}{dt} \circ \psi_\infty\right)(t)\ \frac{d\psi_\infty}{dt} = 1,$$

we obtain that, for any $t \in [0, t_0]$,

$$\Psi\left(\psi_\infty(t)\right) = t. \tag{10}$$

# Proof of Theorem 10

(continuation)

It follows from (9) with $\xi = \psi_\infty(t_0)$, and (10) with $t = t_0$, that

$$\Psi\left(\psi_\infty(t_0)\right) = F(1 + \mathfrak{b}^{-1}\psi_\infty\left(t_0\right)) = \mathfrak{b}^{-1}t_0. \tag{11}$$

Since all the values of $F$ must be contained in the interval $[0, a)$, we deduce from (11) that

$$\mathfrak{b}^{-1}t_0 < a,$$

where $t_0 = K\sigma(x)$. This is equivalent to the necessary condition (2). Finally, we obtain from (11) that

$$\psi_\infty\left(t_0\right) = \mathfrak{b}\left[F^{-1}\left(\mathfrak{b}^{-1}t_0\right) - 1\right].$$

Substituting this into (6), that is $u(x) \geq \psi_\infty\left(t_0\right) + 1$, yields $u(x) \geq \mathfrak{b}\ \left[F^{-1}\left(\mathfrak{b}^{-1}t_0\right) - 1\right] + 1.$ This completes the proof of (1). □

# Nonlinear inequalities $u \geq K(u^q d\sigma) + h$

Let $\sigma \in \mathcal{M}^+(\Omega)$, and let $K$ be a lower semicontinuous kernel. Consider inequalities

$$+\infty > u(x) \geq K(u^q d\sigma)(x) + h(x) \quad d\sigma\text{-a.e. in } \Omega,$$

in the case $q > 0$. Here $h$ is a positive lower semicontinuous function in $\Omega$. In particular, $\inf_F h > 0$ for every compact set $F \subset \Omega$.

We also consider inequalities

$$0 < u(x) \leq -K(u^q d\sigma)(x) + h(x) \quad d\sigma\text{-a.e. in } \Omega,$$

in the case $q < 0$.

We use the notation

$$\Omega' = \{x \in \Omega\colon\; h(x) < +\infty.\}$$

# Nonlinear inequalities $u \geq K(u^q d\sigma) + h$

(continuation)

In most applications, $K = G^\Omega$ is a positive Green's function, and $h$ is a positive superharmonic function, i.e.,

$$h = \mathbf{G}\mu + h_0 > 0, \quad \mu \in \mathcal{M}^+(\Omega), \quad h \geq 0, \quad \Delta h_0 = 0,$$

where $h_0$ is the largest harmonic minorant of $h$.

The case where $h = \text{const} > 0$ was considered above. To treat the general case, along with the kernel $K(x, y)$, we will consider the modified kernel

$$\widetilde{K}(x, y) = \frac{K(x, y)}{h(x)\, h(y)} \quad \text{for } x, y \in \Omega'.$$

Notice that if $+\infty > u \geq K(u^q d\sigma) + h \; d\sigma$-a.e., then obviously

$$\sigma(\Omega \setminus \Omega') = 0.$$

# Domination principle

**Remark**. $\widetilde{K}$ satisfies **(WMP)** in $\Omega'$ provided $K$ satisfies the following weak form of the domination principle **(WDP)** in $\Omega$:

*Given a lower semicontinuous function $h$ in $\Omega$,*

$$K\mu(x) \le M\,h(x), \ \forall\, x \in \mathrm{supp}(\mu) \implies K\mu(x) \le \mathfrak{b}\, M\, h(x), \ \forall\, x \in \Omega$$

*for any compactly supported* $\mu \in \mathcal{M}^+(\Omega)$ such that $K\mu$ is bounded (or for any $\mu$ *with finite energy), and any constant* $M > 0$.

This property is sometimes called a $\mathfrak{b}$-*dilated domination principle*. The classical **domination principle** with $\mathfrak{b} = 1$ holds for Green's kernels $K = G$ associated with a large class of local and non-local operators, and any superharmonic $h > 0$. In the case $h = K\nu + a$ where $\nu \in \mathcal{M}^+(\Omega)$ and $a \ge 0$ is a constant, it is called the *complete maximum principle*.

# Example: quasi-metric kernels

A useful example is given by **quasi-metric kernels** $K$ on $\Omega \times \Omega$ (see [Kalton-Verbitsky 1999], [Hansen 2006], [Frazier-Nazarov-V. 2014]):

$$K(x,y) = \frac{1}{d(x,y)}, \quad x, y \in \Omega,$$

where $d$ is a quasi-metric, i.e., $d\colon \Omega \times \Omega \to [0, +\infty)$, $d \not\equiv 0$, $d(x,y) = d(y,x)$, and there exists a **quasi-metric constant** $\varkappa \geq \frac{1}{2}$ such that the quasi-triangle inequality holds:

$$d(x,y) \leq \varkappa \, [d(x,z) + d(y,z)], \quad \forall \, x, y, z \in \Omega.$$

**Remark.** $d(x,y) \approx \rho(x,y)^\beta$ for some $\beta = \beta(\varkappa)$, where $\rho$ is a metric [Aoki-Rolewicz 1942/57] for **linear** spaces, [Heinonen 2001] in general.

**Lemma (WMP for quasi-metric kernels)**

*Suppose $K$ is a quasi-metric kernel in $\Omega$ with quasi-metric constant $\varkappa$. Then $K$ satisfies the* **(WMP)** *with constant $\mathfrak{b} = 2\varkappa$.*

# Example: Quasi-metric kernels

Many kernels $K$ are quasi-metrically modifiable: the modified kernel $\widetilde{K}(x,y) = \frac{K(x,y)}{h(x)\,h(y)}$ (with some $h > 0$) is quasi-metric (with some modifier $h > 0$). True for $K = G^{\Omega}$ in bounded uniform domains (in particular Lipschitz and NTA domains).

**Lemma (Hansen 2005)**

*Let $\Omega \subset \mathbb{R}^n$ ($n \geq 3$) be a bounded uniform domain (satisfies the interior corkscrew condition and the Harnack chain condition. Define a superharmonic modifier $m(x) = \min[1, \, G^{\Omega}(x, x_0)]$, where $x_0 \in \Omega$ is a fixed pole. Then the modified Green's kernel*

$$\widetilde{G}^{\Omega}(x,y) = \frac{G^{\Omega}(x,y)}{m(x)\,m(y)}, \quad x, y \in \Omega,$$

*is a quasi-metric kernel (with a constant $\varkappa$ independent of $x_0$).*

# Example: quasi-metric kernels

For $w \in \Omega$, let $\Omega_w = \{x \in \Omega\colon \; K(x, w) < +\infty\}$. Then $\widetilde{K}$ is quasi-metric in $\Omega_w$ if $h = K\nu$, where $\nu$ is supported at a single point $w$, i.e., $h(x) = c\, K(x, w)$, $c > 0$. The following lemma yields the **(WDP)** for quasi-metric kernels.

**Lemma (Frazier-Nazarov-Verbitsky 2014)**

*Suppose $K$ is a quasi-metric kernel in $\Omega$ with constant $\varkappa$. Then*

$$K_w(x, y) = \frac{K(x, y)}{K(x, w)\, K(y, w)}, \quad x, y \in \Omega_w,$$

*is a quasi-metric kernel on $\Omega_w$ with quasi-metric constant $4\varkappa^2$.*
*In particular, $K_w$ satisfies the* **(WMP)** *in $\Omega_w$ with constant $\mathfrak{b} = 8\varkappa^3$.*

The lemma follows from the Ptolemy inequality in quasi-metric geometry,

$$d(x, y)\, d(z, w) \le 4\varkappa^2 [d(x, w)\, d(y, z) + d(x, z)\, d(y, w)], \;\; \forall\, x, y, z, w.$$

# Example: quasi-metric kernels

Recall the following

**Lemma (WMP for modified kernels)**

*Suppose* $\boldsymbol{K}$ *is a kernel in* $\boldsymbol{\Omega}$ *which satisfies the* **(WDP)**. *Suppose* $\boldsymbol{h = K\nu \not\equiv +\infty}$ *where* $\boldsymbol{\nu \in \mathcal{M}^+(\Omega)}$. *Then the modified kernel* $\widetilde{\boldsymbol{K}}$ *satisfies the* **(WMP)** *in* $\boldsymbol{\Omega'}$ *with the same constant* $\mathfrak{b}$.
*In particular, if the* **(WDP)** *holds for* $\boldsymbol{K}$ *with* $\mathfrak{b} = \mathbf{1}$, *then* $\widetilde{\boldsymbol{K}}$ *satisfies the strong maximum principle in* $\boldsymbol{\Omega'}$.

**Lemma (WMP for modified quasi-metric kernels)**

*Let* $\boldsymbol{K}$ *be a quasi-metric kernel on* $\boldsymbol{\Omega}$. *Let* $\boldsymbol{h = K\nu}$ *where* $\boldsymbol{\nu \in \mathcal{M}^+(\Omega)}$, $\boldsymbol{h \not\equiv +\infty}$. *Then* $\boldsymbol{K}$ *satisfies the* **(WDP)**, *and* $\widetilde{\boldsymbol{K}}$ *the* **(WMP)** *in* $\boldsymbol{\Omega'}$.

We are now ready to prove Theorem 9 using Theorem 10/11 (in the special case $\boldsymbol{g(t) = t^q}$) and the **(WMP)** for $\widetilde{\boldsymbol{K}}$, or the **(WDP)** for $\boldsymbol{K}$.

# Reduction to the case $h \equiv 1$: Proof of Theorem 9

**Remark.** In the **local case** (Theorems 3-5), we used instead the Doob transform.

Suppose first $q > 0$. Fix $x \in \Omega$ so that $u(x) < \infty$. Then $x \in \Omega'$, i.e., $h(x) < +\infty$, and $d\sigma$-a.e. WLOG we assume $\sigma(\Omega \setminus \Omega') = 0$.
Let $\Omega = \bigcup \Omega_m$ be an exhaustion of $\Omega$: $\Omega_m \uparrow \Omega$ are compact, and $\Omega' = \bigcup \Omega'_m$. Let $d\sigma_m = \chi_{\Omega_m} d\sigma$ where $\mathrm{supp}(\sigma_m) \subseteq \Omega_m$.
Setting

$$v(x) := \frac{u(x)}{h(x)}, \quad x \in \Omega',$$

we see that $v$ satisfies the inequality

$$v(x) \geq \widetilde{K}(v^q d\tilde{\sigma}_m)(x) + 1 \quad d\tilde{\sigma}_m - \text{a.e. in } \Omega_m,$$

where $\tilde{\sigma}_m \in \mathcal{M}^+(\Omega_m)$ is defined by

$$d\tilde{\sigma}_m = h^{1+q}\, d\sigma_m.$$

# Proof of Theorem 9

Notice that $\widetilde{K}$ satisfies the **(WMP)** in $\Omega'$ by the Lemma. By Theorem 10 with $\widetilde{K}$ and $\tilde{\sigma}_m$ in place of $K$ and $\sigma$, it follows that $v$ satisfies the corresponding lower bounds

$$v(x) \geq \left\{ 1 + \mathfrak{b}\Big[ \Big( 1 + \frac{(1-q)\, \widetilde{K}\tilde{\sigma}_m(x)}{\mathfrak{b}} \Big)^{\frac{1}{1-q}} - 1 \Big] \right\}, \quad x \in \Omega_m,$$

where in the case $q > 1$ necessarily

$$\widetilde{K}\tilde{\sigma}_m(x) < \frac{\mathfrak{b}}{q-1}, \quad x \in \Omega_m.$$

Letting $m \to \infty$ we deduce by the monotone convergence theorem

$$v(x) \geq \left\{ 1 + \mathfrak{b}\Big[ \Big( 1 + \frac{(1-q)\, \widetilde{K}\tilde{\sigma}(x)}{\mathfrak{b}} \Big)^{\frac{1}{1-q}} - 1 \Big] \right\}, \quad x \in \Omega,$$

where in the case $q > 1$ necessarily

$$\widetilde{K}\tilde{\sigma}(x) < \frac{\mathfrak{b}}{q-1}, \quad x \in \Omega; \quad d\tilde{\sigma} := h^{1+q}\, d\sigma.$$

# Proof of Theorem 9

Passing back from $(v, \widetilde{K}, \tilde{\sigma})$ to $(u, K, \sigma)$, we deduce the main estimates of Theorem 9 (in the case $q > 0$), provided $K(u^q d\sigma)(x) \le u(x) < \infty$:

$$u(x) \ge h(x) \left\{ 1 + \mathfrak{b} \left[ \left( 1 + \frac{(1-q)\, K(h^q d\sigma)(x)}{\mathfrak{b}\, h(x)} \right)^{\frac{1}{1-q}} - 1 \right] \right\},$$

where in the case $q > 1$ necessarily $h(x) < \infty$ and

$$K(h^q d\sigma)(x) < \frac{\mathfrak{b}}{q-1}\, h(x).$$

Notice that in $K(h^q d\sigma)(x)$ we can integrate over $\Omega$ in place of $\Omega'$ since $\sigma(\Omega \setminus \Omega') = 0$.

In the case $q < 0$, the main estimate and necessary condition of Theorem 9 are deduced in a similar way from Theorem 11 if, for $x \in \Omega$, $0 < h(x) < +\infty$ and $0 < u(x) \le -K(u^q d\sigma)(x) + h(x)$. □

# Some applications to non-local operators, measure coefficients, unbounded solutions

1. ***Convolution equations*** on $\mathbb{R}^n$.

Let $K(x) = k(|x|)$ be an arbitrary radial non-decreasing kernel on $\mathbb{R}^n$. Then $K$ satisfies the **(WMP)** [Ugaheri 1950], and all the estimates hold for positive solutions to the convolution equations with monotone nonlinearity $g : [1, \infty) \to (0, \infty]$,

$$u = k \star g(u^q d\sigma) + 1, \quad q \in \mathbb{R} \setminus \{0\}, \quad \text{on } \mathbb{R}^n,$$

and the homogeneous equation $u = k \star (u^q d\sigma)$ in the sublinear case $g(t) = t^q$, $0 < q < 1$.

2. ***Parabolic equations*** on domains $\Omega$, or Riemannian manifolds,

$$\partial_t u - \Delta u = \sigma u^q + \mu, \quad q \in \mathbb{R} \setminus \{0\}.$$

3. ***Elliptic equations*** with fractional Laplacian on domains $\Omega \subseteq \mathbb{R}^n$, $0 < \alpha < n$, or Riemannian manifolds, with positive Green's function,

$$(-\Delta)^{\frac{\alpha}{2}} u = \sigma u^q + \mu, \quad \forall q \in \mathbb{R} \setminus \{0\}.$$

# Sublinear weighted norm inequalities

Key weighted norm inequalities $K : \mathcal{M}^+(\Omega) \to L^q(\Omega, d\sigma)$ of $(1, q)$-type in the case $0 < q < 1$ (non-classical case):

$$\|K\nu\|_{L^q(\Omega, d\sigma)} \le C\,\|\nu\|, \quad \forall \nu \in \mathcal{M}^+(\Omega), \tag{12}$$

where $\|\nu\|_{\mathcal{M}^+(\Omega)} = \nu(\Omega)$, and $K$ is the integral operator with nonnegative **(WMP)** kernel,

$$K\nu(x) = \int_\Omega K(x, y)\, d\nu(y).$$

Weak-type weighted norm inequalities of $(1, q)$-type, $0 < q \le 1$:

$$\|K\nu\|_{L^{q,\infty}(\Omega, d\sigma)} \le C\,\|\nu\|, \quad \forall \nu \in \mathcal{M}^+(\Omega), \tag{13}$$

are of some interest as well.

# Related sublinear inequalities of $(1, q)$-type

One can use equivalently $(1, q)$-type inequalities with $L^1(\Omega)$ in place of $\mathcal{M}^+(\Omega)$, for kernels $K$ with **(WMP)**:

$$\|Kf\|_{L^q(\Omega, d\sigma)} \le C\, \|f\|_{L^1(\Omega)}, \quad \forall f \in L^1(\Omega). \tag{14}$$

If $K = G^\Omega$ is the Dirichlet Green kernel, then (14) is equivalent to

$$\|\phi\|_{L^q(\Omega, d\sigma)} \le C\, \|\Delta\phi\|_{L^1(\Omega)}, \tag{15}$$

$\forall \phi$ such that $-\Delta\phi \ge 0$ and $\Delta\phi \in L^1(\Omega)$, where $\phi|_{\partial\Omega} = 0$.

Estimate (12), or (15), is key to characterizing all positive weak solutions $u \in L^q_{\mathrm{loc}}(\Omega, \sigma)$ to the sublinear Dirichlet problem $-\Delta u = \sigma\, u^q$.

For finite energy solutions $u \in \dot{W}^{1,2}_0(\Omega)$ we use instead of (15) a $\dot{W}^{1,2}_0(\Omega) \to L^{1+q}(\Omega, d\sigma)$ weighted norm inequality:

$$\|\phi\|_{L^{1+q}(\Omega, d\sigma)} \le C\, \|\nabla\phi\|_{L^2(\Omega, dx)}, \quad \forall \phi \in \dot{W}^{1,2}_0(\Omega).$$

Notice that here again $1 + q < 2$ (non-classical case).

# Sublinear integral equations

The study of $(1, q)$ weighted norm inequalities for for $0 < q < 1$ is motivated by applications to sublinear elliptic PDE of the type

$$\begin{cases} -\Delta u = \sigma\, u^q + \mu \quad \text{in } \Omega, \\ u = \nu \text{ on } \partial\Omega, \end{cases} \qquad \Longleftrightarrow \qquad \begin{cases} u = K(u^q d\sigma) + f \quad \text{in } \Omega, \\ f = K\mu + \mathbf{P}\nu, \end{cases}$$

where $u > 0$; $\mu, \sigma \in \mathcal{M}^+(\Omega)$; $\nu \in \mathcal{M}_b^+(\partial\Omega)$; $P\nu$ harmonic extension. Here $\Omega \subseteq \mathbb{R}^n$ is a domain with non-trivial Green's function $K = G^\Omega$.

The only restrictions imposed on the kernel $K$:

(a) $K$ is quasi-symmetric **(QS)**;

(b) $K$ satisfies the weak maximum principle **(WMP)**.

Here $K$ can be a Green operator associated with $-\Delta$, or a more general elliptic operator, including $(-\Delta)^{\frac{\alpha}{2}}$.

# Conditions on kernels of integral operators

Let $K: \Omega \times \Omega \to [0, +\infty]$ be a nonnegative lower semicontinuous kernel.

**Definition**

A kernel $K$ is quasi-symmetric **(QS)** if there exists a constant $a > 0$ such that

$$a^{-1} K(x, y) \leq K(y, x) \leq a\, K(x, y), \quad x, y \in \Omega. \tag{16}$$

**Definition**

$K \geq 0$ is *degenerate* with respect to $\sigma \in \mathcal{M}^+(\Omega)$ if there exists a set $A \subset \Omega$ with $\sigma(A) > 0$ such that

$$K(\cdot, y) = 0 \quad d\sigma\text{-a.e. } \forall y \in A.$$

Otherwise, $K$ is called non-degenerate with respect to $\sigma$.

See [Sinnamon 2005] in the context of Schur's lemma for positive operators $T: L^p \to L^q$ in the case $1 < q < p$.

# Weak and strong maximum principles

If $\nu \in \mathcal{M}^+(\Omega)$, then by $K\nu$ and $K^*\nu$ we denote the potentials

$$K\nu(x) = \int_\Omega K(x,y)\, d\nu(y), \quad K^*\nu(x) = \int_\Omega K(y,x)\, d\nu(y), \quad x \in \Omega.$$

Recall the following

**Definition**

$K$ satisfies the weak maximum principle **(WMP)** if, for any $\nu \in \mathcal{M}^+(\Omega)$, there exists a constant $\mathfrak{b} \geq 1$ so that

$$K\nu(x) \leq 1, \quad \forall x \in \mathrm{supp}(\nu) \Longrightarrow K\nu(x) \leq \mathfrak{b}, \quad \forall x \in \Omega.$$

If $\mathfrak{b} = 1$, then $K$ satisfies the strong maximum principle **(MP)**.

**Remark.** Green's kernels of many second-order elliptic differential operators are **(QS)** & **(WMP)** [Ancona 2002].

# Potential theory

## Capacities and contents

Let $F \subset X$ be a compact set. For the kernel $K : X \times Y \to [0, +\infty]$, consider several different related notions of **capacity/content:**

$$\mathrm{cap}_0(F) = \sup\Big\{\mu(F)\colon\ \mu \in \mathcal{M}^+(F), \quad K^*\mu(y) \le 1,\ \forall\ y \in Y\Big\},$$
$$\mathrm{cont}(F) = \inf\Big\{\lambda(Y)\colon\ \lambda \in \mathcal{M}^+(Y), \quad K\lambda(x) \ge 1,\ \forall\ x \in F\Big\}.$$

These two notions in fact coincide [Fuglede 1965] via the **Minimax Theorem**. For $X = Y = \Omega$, the Wiener capacity is defined by

$$\mathrm{cap}(F) = \sup\Big\{\mu(F)\colon \mu \in \mathcal{M}^+(F);\ K^*\mu(y) \le 1, \forall\, y \in \mathrm{supp}(\mu)\Big\}.$$

Note that $\mathrm{cap}_0(F) \le \mathrm{cap}(F) \le \mathfrak{b}\, \mathrm{cap}_0(F)$, if $K$ is a **(WMP)** kernel for the upper estimate. The Wiener capacity is most useful if $K$ is **(QS)**.

# Weak-type $(1, q)$-inequality for integral operators

## Theorem 12 (Quinn-Verbitsky 2018)

*Let* $\sigma \in \mathcal{M}^+(\Omega)$, *and* $0 < q \le 1$. *Then the following statements are equivalent:*

1. *There exists a constant* $\varkappa_w > 0$ *such that*

$$\|K\nu\|_{L^{q,\infty}(\sigma)} \le \varkappa_w \|\nu\|, \quad \forall \nu \in \mathcal{M}^+(\Omega).$$

2. *There exists a constant* $c > 0$ *such that*

$$\sigma(F) \le c\Big(\mathrm{cap}_0(F)\Big)^q, \quad \forall \textit{ compact sets } F \subset \Omega.$$

3. *The condition* $K\sigma \in L^{\frac{q}{1-q},\infty}(\sigma)$ *holds (for* $0 < q < 1$*), provided* $K$ *satisfies* **(QS)** *&* **(WMP)**.

**Remark.** Condition (2): V.Maz'ya 1962; if $q > 1$, for quasi-metric kernels enough $\sigma(B(x, r)) \le c\, r^q$; (D.Adams 1972), Riesz kernels.

# Sublinear Schur's Lemma

## Theorem 13 (Quinn-Verbitsky 2018)

*Let* $\sigma \in \mathcal{M}^+(\Omega)$, *and* $0 < q < 1$. *Let* $K \geq 0$ *be a* **(QS)** *&* **(WMP)** *kernel. Then the following statements are equivalent:*

1. *There exists a constant* $\varkappa > 0$ *such that*

$$\|K\nu\|_{L^q(\Omega,\sigma)} \leq \varkappa \, \|\nu\|, \quad \forall \nu \in \mathcal{M}^+(\Omega). \tag{17}$$

2. *There exists a non-trivial supersolution* $u \geq K(u^q d\sigma)$, $u \in L^q(\Omega, d\sigma)$.
3. *There exists a positive solution* $u = K(u^q d\sigma)$, $u \in L^q(\Omega, d\sigma)$, *provided* $K$ *is non-degenerate with respect to* $\sigma$.

**Remarks. 1.** The implication $(1) \Longrightarrow (2)$ in Theorem 13 holds for any $K$.
**2.** The implications (2) or $(3) \Longrightarrow (1)$ generally fail without the **(WMP)**.
**3.** A **minimal** solution $u = \lim u_j$ is constructed ***explicitly*** by iterations:
$u_{j+1} = K(u_j^q d\sigma)$, $u_{j+1} \geq u_j$, $u_0 = c(K\sigma)^{\frac{1}{1-q}}$, $c$ is a small constant.

# Gagliardo's lemma

Sufficiency of (17): The implication $(1) \Longrightarrow (2)$ in Theorem 13 is a special case of Gagliardo's lemma for more general nonlinear maps.

## Lemma (Gagliardo 1965)

*Let* $0 < q < 1$ *and* $\sigma \in \mathcal{M}^+(\Omega)$. *Let* $K \geq 0$ *be a kernel. Suppose the* $(1, q)$*-weighted norm inequality* (17) *holds. Then for every* $\epsilon > 0$, *there is a positive supersolution* $u \in L^q(\Omega, \sigma)$ *such that*

$$u \geq K(u^q d\sigma)$$

*with* $\|u\|^q_{L^q(\Omega,\sigma)} \leq (1+\epsilon)^{\frac{1}{1-q}} \varkappa^{\frac{q}{1-q}}$.

**Remarks. 1.** In general, the Lemma fails if $\epsilon = 0$.
**2.** For non-degenerate $K$, in fact $\epsilon = 0$, and there exists $u = K(u^q \sigma)$.
**3.** The converse fails without the **(WMP)**, even for symmetric positive kernels, for any $\epsilon > 0$.

# Key weak-type $(1, 1)$ lemma

Necessity of (17): To prove $(2) \Longrightarrow (1)$ in Theorem 13, we appeal to Potential Theory. We use some results due to [Fuglede 1960]. Suppose WLOG that $u > 0$ $d\sigma$-a.e., $u \geq K(u^q\sigma)$, and $u \in L^q(\Omega, \sigma)$. We will need the following key weak-type $(1, 1)$-inequality.

**Lemma (Quinn-Verbitsky 2018)**

*Let $K \geq 0$ be a symmetric* **(WMP)** *kernel with constant $\mathfrak{b}$. Suppose $\omega \in \mathcal{M}^+(\Omega)$ is absolutely continuous with respect to the Wiener capacity. Then*

$$\left\| \frac{K\nu}{K\omega} \right\|_{L^{1,\infty}(\Omega,\omega)} \leq \mathfrak{b} \|\nu\|, \quad \forall \nu \in \mathcal{M}^+(\Omega), \tag{18}$$

**Remarks.** 1. In (18) and similar expressions below, we adapt the usual real variables convention $\frac{0}{0} = 0$.
2. The lemma holds for **(QS)** & **(WMP)** kernels with a different constant.

# Proof of the weak-type $(1, 1)$ lemma

**Proof of the lemma:** Let $t > 0$. Define $E_t := \{x \in \Omega : \frac{K\nu(x)}{K\omega(x)} > t\}$. We claim that compact subsets $F \subset E_t$ have finite capacity. This requires that $K(x, x) > 0$ on $E_t$ ($K$ is strictly positive on $E_t$).

Let $A: = \{x \in \Omega : K(x, x) = 0\}$. To verify that $A \cap E_t = \emptyset$, notice that by the **(WMP)**, we have that, for all $x \in A$,

$$K\delta_x(x) = 0 \Longrightarrow K\delta_x(y) = 0, \ \forall y \in \Omega.$$

Thus, $K(x, y) = 0$ on $A \times \Omega$. It follows that, for any $\nu \in \mathcal{M}^+(\Omega)$, $K\nu(x) = 0$ for $x \in A$. Using the convention $\frac{0}{0} = 0$, we see that $\frac{K\nu(x)}{K\omega(x)} = 0$ for all $x \in A$. Hence, $E_t \cap A = \emptyset$ as claimed. This proves that indeed $K(x, x) > 0$ on $E_t$.

# Proof of the weak-type $(1,1)$ lemma

(continuation)

Let $F \subset \Omega$ be a compact set. Assuming that $K(x,x) > 0$ on $F$, by [Fuglede 1960], we can find an equilibrium measure $\mu \in \mathcal{M}^+(F)$ such that $K\mu \geq 1$ q.e. on $F$ and $K\mu \leq 1$ on $\operatorname{supp}(\mu) \subseteq F$.

Thus, if $N := \{x \in F : K\mu(x) < 1\}$, it follows that $\omega(N) = 0$, since $\omega$ is absolutely continuous with respect to capacity.

Moreover, by the **(WMP)**, we have

$$K\mu \leq 1 \text{ on } \operatorname{supp}(\mu) \Longrightarrow K\mu \leq \mathfrak{b} \text{ on } \Omega.$$

From this, since $\frac{K\nu}{t} > K\omega$ on $F$, we deduce the crucial estimate

$$\begin{aligned} \omega(F) &\leq \int_F K\mu \, d\omega = \int_F K\omega_F \, d\mu \\ &\leq \int_F \frac{K\nu}{t} \, d\mu = \frac{1}{t} \int_\Omega K\mu \, d\nu \\ &\leq \frac{1}{t} \int_\Omega \mathfrak{b} \, d\nu = \frac{\mathfrak{b}}{t} \|\nu\|. \end{aligned}$$

# Proof of the weak-type $(1,1)$ lemma
(continuation)

As verified above, on every compact set $F \subset E_t$, the kernel $K$ is *strictly positive*, that is, $K(x,x) > 0$ on $F$. Therefore we have

$$\omega(F) \le \frac{\mathfrak{b}}{t} \|\nu\|.$$

Taking the supremum over all such compact sets $F$, we conclude

$$\omega(E_t) \le \frac{\mathfrak{b}}{t} \|\nu\|,$$

for all $t > 0$, where

$$E_t := \{x \in \Omega \colon \frac{K\nu(x)}{K\omega(x)} > t\}.$$

This establishes the desired weak-type $(1,1)$ estimate. □

# Proof of Theorem 13

## Lemma (infinity sets)

*Let $F$ be a compact set. If $\mu \in \mathcal{M}^+(F)$, $\mu \not\equiv 0$, and $\mathbf{cap}(F) = 0$, then $K^*\mu = +\infty\ d\mu$−a.e in $F$.*

**Proof:** Set

$$E = \{x \in F : \ K^*\mu(x) < +\infty\}.$$

Notice that $E = \bigcup_{n=1}^{\infty} F_n$, where $F_n = \{x \in F : \ K^*\mu(x) \le n\}$ is a closed set by the lower semicontinuity of $K$, and consequently is a compact subset of $F$. In particular, $E$ is a Borel set.
Suppose that $\mathbf{cap}(F) = 0$. Then $\mathbf{cap}(F_n) = 0$, and hence $\mu(F_n) = 0$, for every $n = 1, 2, \ldots$, in view of the definition of $\mathbf{cap}(F_n)$. It follows that

$$\mu(E) \le \sum_{n=1}^{\infty} \mu(F_n) = 0.$$

This proves that $K^*\mu = +\infty\ d\mu$-a.e. on $F$.

# Proof of Theorem 13

(continuation)

## Lemma (absolute continuity w/r to capacity)

*Let $q > 0$. Suppose $\sigma \in \mathcal{M}^+(\Omega)$, and $K^*(u^q \sigma) \le u \; d\sigma$-a.e., where $\int_F u^q \, d\sigma < +\infty$ for every compact set $F \subset \Omega$. Then $d\omega := u^q d\sigma$ is absolutely continuous w/r to capacity: $\mathrm{cap}(F) = 0 \Longrightarrow \omega(F) = 0$. If in addition $u > 0 \; d\sigma$-a.e. on $F$, then $\mathrm{cap}(F) = 0 \Longrightarrow \sigma(F) = 0$.*

**Proof:** Suppose $F$ is a compact set subset of $\Omega$. Since $K^*\omega \le u \quad d\sigma-\text{a.e.}$, we deduce

$$\int_F (K^*\omega)^q \, d\sigma \le \int_F u^q d\sigma = \omega(F) < \infty.$$

Hence $\sigma(\{x \in F : \; K^*\omega = +\infty\}) = 0$. Since $\omega$ is absolutely continuous with respect to $\sigma$, it follows that $\omega(\{x \in F : \; K^*\omega = +\infty\}) = 0$. If $\mathrm{cap}(F) = 0$, then by the previous lemma $\omega(F) = 0$. This clearly yields $\sigma(F) = 0$, unless $u = 0 \; d\sigma$-a.e. on $F$. □

# Proof of Theorem 13

(continuation)

We can now complete the proof of Theorem 13. WLOG we may assume that $K$ is symmetric. Let $u \in L^q(\Omega, \sigma)$ be a positive supersolution, and let $d\omega := u^q \, d\sigma$. By the Lemma, $\omega$ is absolutely continuous with respect to capacity. Suppose $\nu \in \mathcal{M}^+(\Omega)$. If $\nu(\Omega) = +\infty$, there is nothing to prove. In the case that $\nu(\Omega) < +\infty$, we can normalize the measure and assume WLOG that $\nu(\Omega) = 1$.

Since $u$ is a positive supersolution, we have $(K\omega)^q d\sigma \le d\omega$. We estimate, for any $\beta > 0$,

$$\int_\Omega (K\nu)^q \, d\sigma = \int_\Omega \left(\frac{K\nu}{u}\right)^q u^q \, d\sigma \le \int_\Omega \left(\frac{K\nu}{K\omega}\right)^q \, d\omega$$

$$= q \int_0^\beta \omega\left(\left\{x \in \Omega \colon \frac{K\nu(x)}{K\omega(x)} > t\right\}\right) t^{q-1} \, dt$$

$$+ q \int_\beta^\infty \omega\left(\left\{x \in \Omega \colon \frac{K\nu(x)}{K\omega(x)} > t\right\}\right) t^{q-1} \, dt$$

$$= I + II.$$

# Proof of Theorem 13

(continuation)

We first estimate term $\boldsymbol{I}$: clearly, $\boldsymbol{I} \leq q\omega(\Omega) \int_0^{\beta} t^{q-1} dt = \beta^q \, \omega(\Omega)$.
By the key weak-type $(1, 1)$ lemma, we have

$$\omega\left(\left\{x \in \Omega \colon \frac{K\nu(x)}{K\omega(x)} > t\right\}\right) \leq \frac{h \, \nu(\Omega)}{t} = \frac{h}{t}.$$

Consequently, $\boldsymbol{II} \leq \frac{q}{1-q} \mathfrak{b} \beta^{q-1}$. Setting $\beta = \frac{\mathfrak{b}}{\omega(\Omega)}$, we deduce

$$\int_{\Omega} (K\nu)^q \, d\sigma \leq \frac{\mathfrak{b}^q}{1-q} \, \omega(\Omega)^{1-q}.$$

Dropping the restriction $\nu(\Omega) = 1$, and recalling that $d\omega = u^q d\sigma$, we obtain the desired inequality for any $\nu \in \mathcal{M}^+(\Omega)$,

$$\int_{\Omega} (K\nu)^q \, d\sigma \leq \frac{\mathfrak{b}^q}{1-q} \left(\int_{\Omega} u^q \, d\sigma\right)^{1-q} \nu(\Omega)^q.$$

# Proof of Theorem 13

(continuation)

**Remark.** The proof yields that (17) holds with $\varkappa = \frac{\mathfrak{b}}{(1-q)^{\frac{1}{q}}} \|u\|^{1-q}_{L^q(\Omega,\sigma)}$ for **symmetric** kernels $K$. For **(QS)** kernels, we use a symmetrized kernel $\frac{K+K^*}{2}$ to deduce a similar estimate where $\varkappa$ depends also on the quasi-symmetric constant $a > 0$ in condition (16).

□

In the next lemma, we give some sufficient/necessary conditions for $\varkappa < \infty$ in (17) in terms of Lorentz spaces $L^{s,r}(\Omega, \sigma)$ with quasi-norm

$$\|f\|^r_{L^{s,r}(\Omega,\sigma)} = s \int_0^\infty [t^s \sigma(x \in \Omega \colon |f(x)| > t)]^{\frac{r}{s}} \frac{dt}{t} < \infty.$$

Here $L^{s,s}(\Omega, \sigma) = L^s(\Omega, \sigma)$ and $L^{s,\infty}(\Omega, \sigma)$ is the weak $L^s$ space.

## Sufficient/necessary conditions for the $(1, q)$-inequality

**Lemma (Quinn-Verbitsky 2018)**

*Let $\sigma \in \mathcal{M}^+(\Omega)$ and $0 < q < 1$. If $K$ satisfies* **(QS)** *&* **(WMP)**, *then the $(1, q)$-weighted norm inequality* (17) *holds if $K\sigma \in L^{\frac{q}{1-q}, q}(\Omega, \sigma)$. Conversely, if (1) holds, then $K\sigma \in L^{\frac{q}{1-q}}(\Omega, \sigma)$.*

**Remarks. 1.** The exponents $\frac{q}{1-q}$ and $q$ are sharp: inequality (17) may fail if $K\sigma \in L^{s,r}(\Omega, \sigma)$ with $s = \frac{q}{1-q}$ and $r > q$, or $0 < s < \frac{q}{1-q}$, $r > 0$.
**2.** The condition $K\sigma \in L^{s,r}(\Omega, \sigma)$ with $s = \frac{q}{1-q}$ and $r < q$ is not necessary.
**3.** Another (independent) ***necessary*** condition is

$$\sup_{x \in \Omega} \int_\Omega K(x, y)^q d\sigma(y) < \infty.$$

# Necessary condition for the $(1, q)$-inequality

**Remark.** The necessity of the condition $\int_\Omega (K\sigma)^{\frac{q}{1-q}} d\sigma < \infty$ for the existence of a nontrivial supersolution

$$u(x) \geq K(u^q d\sigma)(x), \quad u \in L^q(\Omega, \sigma),$$

for **(WMP)**-kernels $K$, is immediate from Theorem 8 proved above:

**Theorem 8 (Grigor'yan-Verbitsky 2020)**

*Suppose $K$ is a positive kernel on $\Omega$ satisfying the* **(WMP)** *with constant $\mathfrak{b} > 0$. Let $0 < q < 1$. If $u \geq 0$ is a non-trivial supersolution, then*

$$u(x) \geq \mathfrak{b}^{-\frac{q}{1-q}} (1-q)^{\frac{1}{1-q}} \Big[ K\sigma(x) \Big]^{\frac{1}{1-q}} \quad d\sigma\text{—a.e. in } \Omega.$$