POTENTIAL THEORY AND NONLINEAR ELLIPTIC EQUATIONS

1. Introduction

This minicourse will be concerned with certain classes of nonlinear elliptic equations, in particular, quasilinear equations of p-Laplace type with nonlinear lower-order terms in a domain $\Omega \subseteq \mathbb{R}^n$ or a Riemannian manifold:

$$-\mathrm{div}\mathcal{A}(x, \nabla u) = f(x, u, \nabla u) \quad \text{in } \Omega. \tag{1.1}$$

Here $\mathrm{div}\mathcal{A}$ is the so-called $\mathcal{A}$-Laplacian, where $\mathcal{A} : \Omega \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is a nonlinear measurable function subject to standard growth and monotonicity assumptions of order $p \in (1, \infty)$.

A model example is the p-Laplace operator $\Delta_p u = \mathrm{div}(\nabla u\, |\nabla u|^{p-2})$. The linear case $p = 2$ corresponds to the classical Laplace operator Δ.

We will discuss existence and regularity properties, as well as sharp pointwise and integral estimates of solutions. Various classes of solutions, including weak solutions, and solutions in Lebesgue spaces L^r, Sobolev spaces $W^{\alpha,r}$ and BMO spaces will be treated.

We intend to study in detail positive solutions u to equation (1.1) with right-hand sides of the type $f(x, u) = \sigma u^q + \mu$, that is, the equation

$$-\mathrm{div}\mathcal{A}(x, \nabla u) = \sigma u^q + \mu \quad \text{in } \Omega. \tag{1.2}$$

Here μ, σ are measurable functions (or measures) in Ω, and $q \in \mathbb{R}\backslash\{0\}$.

In the case where $p = 2$ and $\mathcal{A}(x, \xi)$ is a linear matrix function, the $\mathcal{A}$-Laplacian is a linear uniformly elliptic second-order differential operator with bounded measurable coefficients. A related integral equation

$$u = \mathbf{G}(\sigma u^q) + \mathbf{G}\mu, \tag{1.3}$$

where $\mathbf{G}$ is Green's operator for $\mathrm{div}\mathcal{A}$, or a nonlocal operator with positive Green's kernel, will be treated (Grigor'yan and Verbitsky, 2020).

If $f \geq 0$, then the right-hand side of (1.1) is a nonlinear source term, and it is natural to consider $\mathcal{A}$-*superharmonic* solutions u to (1.1). A nonlinear potential theory for the equation

$$-\mathrm{div}\mathcal{A}(x, \nabla u) = \mu, \tag{1.4}$$

for positive measures μ, was developed by Kilpeläinen and Malý, 1994. They obtained bilateral pointwise estimates of positive solutions u to (1.4) in terms of *Wolff potentials*.

In this theory, the p-capacity plays a fundamental role. It is defined, for a compact set $E \subset \Omega$ by

$$\mathrm{cap}_p(E) = \inf \left\{ \|\nabla u\|^p_{L^p(\Omega)} \colon u \geq 1 \ \text{ on } \ E,\ u \in C_0^\infty(\Omega) \right\}.$$

In our study of equations (1.2), we will also use more general capacities associated with Sobolev spaces $W^{\alpha,r}(\Omega)$, where α, r may depend on p, q (Phuc and Verbitsky, 2008, 2009).

Geometric fully nonlinear equations with $F_k(u)$ in place of $\mathrm{div}\mathcal{A}(x, \nabla u)$, where F_k is the k-Hessian operator, are also of great interest.

The fully nonlinear k-Hessian operator F_k ($k = 1, 2, \ldots, n$) is defined by

$$F_k(u) = \sum_{1 \le i_1 < \cdots < i_k \le n} \lambda_{i_1} \cdots \lambda_{i_k}, \tag{1.5}$$

where $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of the Hessian matrix $D^2 u$. It is known that F_k is elliptic on the cone of k-convex functions. Here $k = 1$ corresponds to the classical Laplacian, and $k = n$ to the Monge-Ampere operator (convex functions).

A nonlinear potential theory for such equations was developed by Trudinger and Wang, 1999, 2002 and Labutin, 2003. It uses heavily the notion of the k-capacity, which turned out to be equivalent to the fractional Sobolev capacity (Phuc and Verbitsky, 2008), along with nonlinear Hessian Sobolev and Hessian Poincaré inequalities of Trudinger and Wang (new proofs and extensions were given in Verbitsky, 2015).

We will also consider another model equation of the type (1.1), namely,

$$-\Delta_p u = \sigma \frac{|\nabla u|^q}{u^\gamma} + \mu \quad \text{in } \Omega. \tag{1.6}$$

Even for $\sigma = \text{const}$, this is a challenging problem. We will treat the special case with *singular natural growth* in the gradient ($q = p$, $\gamma = 1$) and constant $\sigma = b > 0$ (Cao and Verbitsky, 2017).

In the case $p = 2$ and $q = 0$, $\gamma > 0$, sharp estimates of solutions u for σ changing sign, have recently been obtained by Grigor'yan and Verbitsky, 2019.

Another important special case is $\gamma = 0$. For $p = 2$ and $\sigma = \text{const}$, sharp results were obtained by Hansson, Maz'ya and Verbitsky, 1999. For $p \neq 2$, the case $q = p$ (natural growth in the gradient) was considered by Jaye and Verbitsky, 2012. In the case $q \neq p$, $\sigma = \text{const}$, such equations are currently a subject of extensive studies (Nguyen Cong Phuc and his co-authors).

Pointwise *gradient* estimates of $|\nabla u|$ are important for equations such as (1.2) and (1.6). Such gradient estimate in terms of Wolff and Riesz potentials for equations (1.4) have been obtained recently (Mingione and his co-authors). Combining them with certain weighted norm inequalities makes it possible to study various classes of solutions to equations (1.2), including solutions in BMO spaces (Phuc and Verbitsky, 2021).

The potential theory approach is is useful in studies of other important nonlinear equations and systems, as well as their analogues on Riemannian manifolds (Grigor'yan, Sun, and Verbitsky, 2020).